University of California
Los Angeles

Fourier Coefficients of Triangle Functions

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

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2008

The dissertation of John Garrett Leo is approved.

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2008

to Kyoko and *Kyle*

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Acknowledgments

The three mathematics department members of my thesis committee were not only the first professors I met at UCLA but also the most influential and helpful to me here. First and foremost I want to thank my thesis advisor, Bill Duke. Bill not only suggested looking into the problem that led into this thesis, but provided invaluable guidance throughout. In addition he taught me how to approach mathematics and research, and did everything in his power to help make me successful. I cannot imagine a better advisor.

Don Blasius has been another wonderful mentor to me, both informally and through the courses he taught and participating seminars he ran, and also was my official mentor for the first calculus class I taught. I learned a great deal from him as well.

Richard Elman first taught me all of the linear algebra I'd forgotten, and then was my professor for both the graduate algebra sequence I took and the honors undergraduate algebra sequence I TAed twice. This is by far the best run and most successful math course I have ever experienced and has become a model for how I wish to run my own courses. His door is always open, and I have taken advantage of many opportunities to visit and talk.

Thanks to Jim Ralston for organizing the math department hikes, and even more the Sierra backpacking trips. It's been a pleasure to hike and climb with him and I hope we will continue even after I graduate.

I returned to graduate school because I wanted to teach, and was lucky to be able to teach some superb classes here that confirmed my decision. I want to thank the many wonderful students I've had the pleasure to interact with, especially in linear algebra (the very first course I TAed), honors calculus, and most of all the two years of honors algebra.

Lora Danley was my inspiration to return to graduate school and pursue a teaching career. Betsy James provided valuable career advice near the end of my time here. My fellow classmates at UCLA taught me a great deal and made my time here a pleasure. The math department in general has been a friendly and wonderful place to study. Thanks also to my friends in the Bay Area and Colorado for keeping in touch and getting together every summer.

Music of China has been my most important experience outside of math at UCLA, and I thank Chi Li both for running the ensemble and for so much personal help. I'm very happy to have learned to play some beautiful music and also to meet so many terrific people in this ensemble.

I want to especially thank my wife Kyoko and son Kyle, for creating a wonderful home and family life that has been my refuge from school for the last six years.

Finally I want to thank my best friends at UCLA, particularly Kaya Yuki and Lydia Yue. If it weren't for you I might have graduated earlier, but I wouldn't have enjoyed the experience nearly as much.

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Abstract of the Dissertation

Fourier Coefficients of Triangle Functions

by

John Garrett Leo

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2008

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Triangle functions J_m are generalizations of the j modular function that map the interior of a hyperbolic triangle with vertices $(i, -\exp(-\pi i/m), i\infty)$ to the upper half plane, where $m \geq 3$ is an integer. The corresponding groups, generalizing the modular group, are known as Hecke groups and are generated by $S(z) = -1/z$ and $T_m(z) = 2\cos(\pi/m)$.

Fourier coefficients of the J_m were first studied by Lehner in 1954, but were shown to be transcendental (except in the cases $m = 3, 4, 6, \infty$) by Wolfart in 1981 and research on them mostly stopped. However the transcendental part is easily factored out, and the remaining part, a rational integer, has very interesting properties, especially with respect to which primes divide the denominator. In this thesis experimental evidence, a conjecture, and a proof of part of the conjecture are presented.

CHAPTER 1

Introduction

This thesis concerns the number-theoretic properties of fourier coefficients of triangle functions. Triangle functions are conformal mappings from hyperbolic triangles to the upper half plane, generalizing the modular function j. Modular functions, along with related functions known as modular forms and automorphic forms, are principal objects of study in number theory, and most famously played a major role in Wiles' proof of Fermat's Last Theorem.

A modular form (of weight k) is a holomorphic function on the upper half plane that satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ we have $f(z+1) = f(z)$ and so f is periodic and has a fourier expansion. We require the function to be "holomorphic at infinity", which means the expansion has no negative powers. In other words, letting $q = e^{2\pi i z}$, we can write

$$f(z) = \sum_{n=0}^{\infty} a_n q^n.$$

If $a_0 = 0$ then f is called a cusp form. For a given weight k the modular forms and cusp forms each form finite-dimensional vector spaces, and linear operators known as Hecke operators act on these spaces. The functions thus have a great deal of structure and one is able to study their properties using linear algebra and other tools.

The canonical example of a modular form is the Eisenstein series, which for even weight $k \geq 4$ can be defined as

$$E_k(z) = \frac{1}{2\zeta(k)} \sum_{c,d \in \mathbb{Z}}{}' (cz+d)^{-k}$$

where $\sum'$ means to exclude the case in which both $c = 0$ and $d = 0$ and $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$ is the Riemann zeta function. The Eisenstein series has a fourier expansion

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where B_k are the Bernoulli numbers (rational integers) defined by setting

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$$

and $\sigma_k(n) = \sum_{d|n, d>0} d^k$. That this sum of divisors function should appear in the fourier coefficients already gives the Eisenstein series number theoretic importance, but far more can be done. One remarkable application of Eisenstein series (defined more generally on "congruence subgroups" of $SL_2(\mathbb{Z})$) is to compute the number of ways a positive integer can be written as the sum of k squares, for k even (see for example [DS05], section 1.2).

The canonical cusp form is the discriminant modular form

$$\Delta(z) = g_2(z)^3 - 27 g_3(z)^2,$$

where $g_2(z) = \frac{(2\pi)^4}{12} E_4(z)$ and $g_3(z) = \frac{(2\pi)^6}{216} E_6(z)$. It has weight 12 and the fourier expansion

$$\Delta(z) = (2\pi)^{12} \sum_{k=1}^{\infty} \tau(k) q^k$$

where $\tau(n)$ is the Ramanujan tau function, another important function in number theory. It can be shown using Hecke operators that $\tau(k)$ is multiplicative; that is, that $\tau(mn) = \tau(m)\tau(n)$ if m and n are relatively prime. The discriminant modular form also has a remarkable product formula

$$\Delta(z) = (2\pi)^{12} q \prod_{k=1}^{\infty} (1 - q^n)^{24}.$$

A modular function is similar to a modular form save that it can be meromorphic on the upper half plane and at infinity (the fourier series can have finitely many negative powers); on the other hand it must satisfy the stronger condition

$$f\left(\frac{az+b}{cz+d}\right) = f(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. The canonical modular function is the j function which can be defined as

$$j(z) = \frac{1728 g_2(z)^3}{\Delta(z)} = 1728 \frac{E_4(z)^3}{E_4(z)^3 - E_6(z)^2}.$$

The fourier series for this function has all integral coefficients and begins

$$j(z) = q^{-1} + \sum_{k=0}^{\infty} a_n q^n = q^{-1} + 744 + 196884q + 21493760q^2 + 8642909970q^3 + \cdots.$$

Lehner ([Leh49a, Leh49b]) proved several congruence relations on these coefficients, for example that $a_{2k} \equiv 0 \pmod{2^{11}}$ and $a_{3k} \equiv 0 \pmod{3^5}$ for all k (for an exposition see [Apo90], chapter 4).

More remarkably, around 1979 John McKay noticed a simple relationship between these coefficients and the dimensions of the smallest irreducible representations of the Monster simple group. This connection was dubbed "Monstrous Moonshine" by Conway and Norton ([CN79]), and was proven by Borcherds in 1992 ([Bor92]). For an overview see [Gan04].

One can generalize the notions of modular forms and functions by replacing $SL_2(\mathbb{Z})$ by some other group. One important class is that of congruence subgroups, that is subgroups of $SL_2(\mathbb{Z})$ in which the matrix coefficients satisfy certain congruence conditions. Another class, far less studied, is the class of Hecke groups G_m, subgroups of $SL_2(\mathbb{R})$ generated by the two matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T_m = \begin{pmatrix} 1 & \lambda_m \\ 0 & 1 \end{pmatrix}$$

where $m \geq 3$ (or $m = \infty$) and $\lambda_m = 2\cos(\pi/m)$. The group G_3 is simply the modular group $SL_2(\mathbb{Z})$. These groups were studied by Hecke ([Hec36, Hec38]) and also are special cases of larger classes of groups known as triangle groups and more generally Fuchsian groups that have been studied since the 19th century, and are still important not only in number theory, but also for example the Geometrization Program of Thurston in topology ([MR03]).

The quotient of the upper half plane by the action of the group is a hyperbolic triangle, and we are interested in modular functions invariant under this action. These functions can be viewed as conformal mappings from the interior of a hyperbolic triangle (with vertices at i, $-e^{-\pi i/m}$, and $i\infty$) to the upper half plane. It can be shown that any such function is a rational function of a specific function $J_m(z)$, whose inverse (mapping the upper half plane to the triangle) is called a Schwarz triangle function. The modular function j for $m = 3$ satisfies $j(z) = 1728J_3(z)$.

The standard way to determine the coefficients of j and their properties is to first compute the coefficients of the Eisenstein series, using either a derivation from the product formula for sine or a more natural derivation using double cosets (see [Leh64], p. 282 and [Iwa97], Chapter 3), and then derive the coefficients of j from that. However for the Hecke groups it is also possible derive the fourier series for J_m as the inverse of the fourier series for the corresponding Schwarz triangle function, which can be computed explicitly as a ratio of solutions to a specific hypergeometric differential equation (see for example [Neh52], Chapters V and VI, or [Car60], Chapter Two). One can then in turn derive the coefficients of the corresponding Eisenstein series and discriminant modular form from that of J_m.

Lehner ([Leh54]) was the first to take this approach, and his work was slightly generalized and clarified by Raleigh ([Ral63]). Let

$$J_m(z) = \sum_{n=-1}^{\infty} a_n q^n$$

where $q = e^{2\pi i z/\lambda_m}$ and $\lambda_m = 2\cos\frac{\pi}{m}$. Raleigh proved that $\log a_{-1} = -2\psi(1) + \psi(1-\alpha) + \psi(1-\beta) - \pi\sec(\pi/m)$ (where $\psi(z) = \Gamma'(z)/\Gamma(z)$) and that $a_{-1} \in \mathbb{Q}$ for $m = 3, 4, 6$ and ∞; Wolfart ([Wol81]) proved that a_{-1} is transcendental for all other values of m. Raleigh further proved that for $n \geq 0$

$$a_n = \frac{P_n(m^2)}{Q_n a_{-1}^n m^{2n+2}},$$

where P_n is an integer polynomial of degree $n+1$ and Q_n is an integer. In other words, save for a power of a_{-1}, the coefficients are all rational numbers. Lehner had proven this earlier using a different method.

Akiyama ([Aki92]) proved Raleigh's conjecture that for $n \geq 2$, the numerator of a_n is divisible by $m^2 - 4$, and also that the primes dividing Q_n are all less than or equal to $n+1$. He also examined divisibility of coefficients by powers of 2 and 3 ([Aki93]). No further work seems to have been done on number-theoretic properties of the fourier coefficients in the cases other than $m = 3, 4, 6$ and ∞. Perhaps the reason is that the coefficients are transcendental in these cases, even though the transcendental part can easily be factored out. Also Takeuchi ([Tak77]) proved that the Hecke groups are arithmetic precisely in the cases $m = 3, 4, 6$ and ∞. For the non-arithmetic cases it appears Hecke operators and other machinery used to prove congruence and other results can no longer be used.

In this thesis, based upon empirical evidence, we present the following conjecture.

Conjecture. *Let $m = 5$ or $m \geq 7$. Let $J_m(z) = \sum_{n=-1}^{\infty} a_n q^n$, and let*

$$a_n = \frac{C_n}{D_n a_{-1}^n 2^{6n+6} m^{3n+3}}$$

where $C_n, D_n \in \mathbb{Z}$ and $\gcd(C_n, D_n) = 1$. Then the primes dividing $\{D_n : n \geq 1\}$ are $\{p : p \nmid 2m$ and $p \not\equiv \pm 1 \pmod{m}\}$. Furthermore if n_0 is the least n for which p divides the denominator of D_n, then $n_0 = p^k - 1$ for some $k \geq 1$.

Using methods devised by Dwork ([Dwo69, Dwo73]) to prove integrality results for power series solutions to certain generalized hypergeometric differential equations, we prove the following special case of the conjecture.

Theorem. *Let $m \geq 3$. In the notation of the conjecture above, p does not divide D_n for all odd primes $p \equiv 1 \pmod{4m}$.*

It appears we should be able to strengthen this to prove that p does not divide D_n for all odd primes $p \equiv \pm 1 \pmod{m}$ which essentially proves one direction of the conjecture.

This result is interesting because it shows that the coefficients are p-adic integers for these primes, and therefore may satisfy further congruence conditions in this context. Furthermore the primes that appear in the denominators do not split in the ring of integers $\mathbb{Z}[\lambda_m]$ of $\mathbb{Q}(\lambda_m)$, which is in turn the maximal real subfield of the cyclotomic field $\mathbb{Q}(e^{\pi i/m})$. This suggests that this ring of integers is the proper context in which to study the properties of the coefficients.

Although no further work on the properties of the fourier coefficients seems to have been done since Akiyama, both triangle functions and Hecke groups have continued to be active areas of research. We discuss this research and its connection with our results in the final chapter.

The thesis is structured as follows. Chapter 2 describes the derivation of the triangle functions and Hecke Groups. Chapter 3 contains the main results of the thesis. After rederivation of some previous results, computer experiments are presented along with the conjecture they led to, and the special case of the conjecture is proven. Chapter 4 discusses modular forms, both what information can be determined about their coefficients from the J_m, and attempts to derive the coefficients directly. These latter attempts run into obstacles, which are thus circumvented by deriving coefficient information from the J_m functions. Finally Chapter 5 presents the conclusion and future work.

CHAPTER 2

Triangle Functions

In this chapter we describe and rederive major known results, all classical, regarding triangle functions. We first show how they are defined as biholomorphic functions mapping hyperbolic triangles to the upper half plane. We then define Hecke groups and demonstrate their relation to triangle functions.

2.1 Mappings of Hyperbolic Triangles

The Riemann Mapping Theorem states that there exists a conformal (biholomorphic) map from any simply-connected non-empty proper subset of $\mathbb{C}$ to the upper half plane $\mathbb{H}$. It does not, however, describe how to derive such functions. For certain classes of regions explicit functions can be determined. One class is the interiors of polygons; the mapping from $\mathbb{H}$ to the interior of the polygon is an example of an elliptic integral, and its inverse (from the interior of the polygon to $\mathbb{H}$) is an elliptic function.

Another class is the interiors of polygons with curvilinear edges, and in particular hyperbolic triangles, which we concern ourselves with here. In this case the mapping from $\mathbb{H}$ to the interior of the triangle is a ratio of linearly independent solutions to the hyperbolic differential equation, and the inverse is an automorphic function generalizing the Klein j-invariant (which corresponds to the case in which the three angles are $0, \frac{\pi}{2}, \frac{\pi}{3}$).

For references for this material see Stein and Shakarchi ([SS03], Chapter 8), Nehari ([Neh52], Chapters V and VI), Caratheodory ([Car60], Chapter Two) and Fricke ([Fri30], pp. 105–115). The exposition here is derived primarily from Nehari and Caratheodory.

Let $f(z)$ be a biholomorphic map from $\mathbb{H}$ to the interior of a hyperbolic triangle whose sides are circular arcs, whose vertices are A, B, C, and whose angles at those vertices are $\pi\alpha, \pi\beta, \pi\gamma$, where α, β, γ are all non-negative and $\alpha+\beta+\gamma < 1$. We wish to derive a formula for f. To this end we use the Schwarzian derivative, defined to be

$$\{w, z\} = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 \tag{2.1}$$

where w is a function of z. The Schwarzian derivative has the property that it is invariant under fractional linear transformations. That is, if

$$W(z) = \frac{aw(z) + b}{cw(z) + d}$$

where $ad - bc \neq 0$, then $\{W, z\} = \{w, z\}$. The proof is a straightforward calculation. First take the derivative of both sides to get

$$W' = \frac{ad - bc}{(cw + d)^2} w'$$

and then take the logarithmic derivative to get

$$\frac{W''}{W'} = \frac{w''}{w'} - \frac{2cw'}{cw + d}.$$

Taking another derivative gives

$$\left(\frac{W''}{W'}\right)' = \left(\frac{w''}{w'}\right)' + \frac{2c^2 w'^2}{(cw + d)^2} - \frac{2cw''}{cw + d}$$

and alternatively squaring gives

$$\left(\frac{W''}{W'}\right)^2 = \left(\frac{w''}{w'}\right)^2 + \frac{4c^2 w'^2}{(cw + d)^2} - \frac{4cw''}{cw + d}.$$

Subtracting half the second equation from the first gives $\{W, z\} = \{w, z\}$.

Now without loss of generality (compose with a fractional linear transformation if necessary), assume f maps 0 to A, 1 to B and ∞ to C. Assume $\alpha > 0$. Let g_A be the fractional linear transformation that maps A to the origin O, B to point X on the positive real axis, and C to point Z so that the circular arc from A to C is mapped to a straight line and so that the angle ZOX is $\pi\alpha$. Let $f_A = g_A \circ f$. Then $\{f_A, z\} = \{f, z\}$. If we consider the function $h_A(z) = f_A^{1/\alpha}(z)$, then h_A is holomorphic is the upper half-plane and maps a segment of the real axis centered on the origin to a segment of the real axis (with 0 mapping to 0) and so by Schwarz reflection is holomorphic at 0. Also h_A is conformal at 0 and therefore $h'_A(0) \neq 0$. Letting $h_A(z) = zf_1(z)^{1/\alpha}$ it follows we can write $f_A(z) = z^\alpha f_1(z)$ where $f_1(z)$ is holomorphic at $z = 0$, $f_1(0) \neq 0$, and $f_1(z)$ is real when z is real.

If $\alpha = 0$, then instead let g_A be the fractional linear transformation mapping A to $-\infty$, B to a point X on the real axis, and Z to a point along the line through $i\pi$ parallel to the real axis. Again let $f_A = g_A \circ f$, and now consider $h_A(z) = e^{f_A(z)}$. Then as before h_A is conformal at 0 and letting $h_A(z) = ze^{f_1(z)}$ we can write $f_A(z) = \log z + f_1(z)$ where $f_1(z)$ is holomorphic at $z = 0$, $f_1(0) \neq 0$, and $f_1(z)$ is real when z is real.

In either case we compute

$$\{f, z\} = \{f_A, z\} = \frac{1 - \alpha^2}{2z^2} + \frac{c_A}{z} + F_A(z) \tag{2.2}$$

where c_A is a real constant and $F_A(z)$ is holomorphic.

Similarly for $\beta > 0$ let g_B be the fractional linear transformation that maps B to the point $P = 1$, C to point X on the positive real axis, and D to point Z so that the circular arc from B to A is mapped to a straight line and so that the angle ZPX is $\pi\beta$. Let $f_B = g_B \circ f$. Then $\{f_B, z\} = \{f, z\}$. Letting

$h_B(z) = f_B^{1/\beta}(z) = (z-1)f_2(z)^{1/\beta}$, as before we have $f_B(z) = (z-1)^\beta f_2(z)$ where $f_2(z)$ is holomorphic at $z = 0$, $f_2(0) \neq 0$, and $f_2(z)$ is real when z is real. In the case $\beta = 0$ we get $f_B(z) = \log(z-1) + f_2(z)$ using the analogous procedure to the case $\alpha = 0$.

We compute

$$\{f, z\} = \{f_B, z\} = \frac{1-\beta^2}{2(1-z)^2} + \frac{c_B}{1-z} + F_B(z)$$

where c_B is a real constant and $F_B(z)$ is holomorphic.

To handle the case at ∞ use the substitution $z = 1/t$ and note that by (2.2) we have

$$\{f, t\} = \frac{1-\gamma^2}{2t^2} + \frac{c_3}{t} + F_3(t).$$

A straightforward calculation shows that $\{f, z\} = t^4 \{f, t\}$, from which it follows

$$\{f, z\} = \frac{1-\gamma^2}{2z^2} + \frac{c_C}{z^3} + \frac{1}{z^4} F_C(1/z). \tag{2.3}$$

It follows

$$\{f, z\} - \frac{1-\alpha^2}{2z^2} - \frac{c_A}{z} - \frac{1-\beta^2}{2(1-z)^2} - \frac{c_B}{1-z}$$

is entire and converges to zero as $z \to \infty$ (therefore is also bounded), and so by Liouville's theorem must be identically zero. Thus

$$\{f, z\} = \frac{1-\alpha^2}{2z^2} + \frac{c_A}{z} + \frac{1-\beta^2}{2(1-z)^2} + \frac{c_B}{1-z}. \tag{2.4}$$

Furthermore from (2.3) it follows

$$\lim_{z\to\infty} z\{f, z\} = 0, \quad \lim_{z\to\infty} z^2 \{f, z\} = \frac{1-\gamma^2}{2},$$

and combining these with (2.4) yields the relations $c_A = c_B$ and

$$\frac{1-\gamma^2}{2} = \frac{1-\alpha^2}{2} + \frac{1-\beta^2}{2} - c_A,$$

from which we ultimately obtain

$$\{f,z\} = \frac{1-\alpha^2}{2z^2} + + \frac{1-\beta^2}{2(1-z)^2} + \frac{1-\alpha^2-\beta^2+\gamma^2}{2z(1-z)}. \tag{2.5}$$

We now wish to solve this differential equation. By the definition of Schwarzian derivative, it is a third-order equation, but it suffices to solve a second-order equation.

Lemma 1. *Let*

$$u''(z) + p(z)u(z) = 0$$

be a linear second-order homogeneous differential equation, and let $u_1(z)$ *and* $u_2(z)$ *be two linearly independent solutions. Then*

$$w(z) = \frac{u_1(z)}{u_2(z)}$$

is a solution of the equation

$$\{w,z\} = 2p(z).$$

Proof. We have $u_1'' + pu_1 = 0$, $u_2'' + pu_2 = 0$, and since $u_1 = u_2 w$ we also have $u_2 w'' + 2u_2' w' = 0$. Therefore

$$\frac{w''}{w'} = -2\frac{u_2'}{u_2}$$

and

$$\left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 = -2\left(\frac{u_2'}{u_2}\right)' - 2\left(\frac{u_2'}{u_2}\right)^2 = -2\frac{u_2''}{u_2}$$

from which the result follows. □

By the lemma it suffices to find two linearly independent solutions $u_1(z), u_2(z)$ to the differential equation

$$u''(z) + \frac{1}{4}\left[\frac{1-\alpha^2}{z^2} + \frac{1-\beta^2}{(1-z)^2} + \frac{1-\alpha^2-\beta^2+\gamma^2}{z(1-z)}\right]u(z) = 0. \tag{2.6}$$

However since we only need the ratio of the two solutions, we can further simplify, and show that the ratio $y_1(z)/y_2(z)$ of two solutions of a hypergeometric equation will be equal to $u_1(z)/u_2(z)$. To do this we examine the equation

$$y'' + P(z)y + Q(z)y = 0 \tag{2.7}$$

where $y(z) = \sigma(z)u(z)$, P and Q are to be determined, and

$$\sigma(z) = e^{-\frac{1}{2}\int P(z)\,dz}.$$

Note that if $y_1(z)$ and $y_2(z)$ are two linearly independent solutions of (2.7), then by definition $y_1/y_2 = u_1/u_2$.

Substituting $y(z) = \sigma(z)u(z)$ into (2.7), we get

$$u'' + \left(Q - \frac{1}{4}P^2 - \frac{1}{2}P'\right)u = 0$$

and by comparison with (2.6) we want

$$Q - \frac{1}{4}P^2 - \frac{1}{2}P' = \frac{1}{4}\left[\frac{1-\alpha^2}{z^2} + \frac{1-\beta^2}{(1-z)^2} + \frac{1-\alpha^2-\beta^2+\gamma^2}{z(1-z)}\right].$$

This equation holds if we let

$$P(z) = \frac{c-(a+b+1)z}{z(1-z)}, \quad Q(z) = -\frac{ab}{z(1-z)}$$

where

$$a = \frac{1}{2}(1-\alpha-\beta+\gamma), \quad b = \frac{1}{2}(1-\alpha-\beta-\gamma), \quad c = 1-\alpha.$$

With these values equation (2.7) becomes

$$z(1-z)y'' + [c-(a+b+1)z]y' - aby = 0 \tag{2.8}$$

which is the well-known hypergeometric differential equation. Details of its properties and solutions can be found in for example Whittaker and Watson ([WW96], Chapter XIV).

One solution to (2.8) is the hypergeometric series

$$F(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} z^k, \tag{2.9}$$

as can be verified by direct computation. Here $(a)_k$ is the rising factorial with k terms:

$$(a)_k = a(a+1)\cdots(a+k-1).$$

There are several ways to calculate a linearly independent second solution to the hypergeometric differential equations, depending upon the constants chosen. In the case we are interested in, one of the angles, say $\pi\alpha$, will be zero. In this case we have

$$a = \frac{1}{2}(1-\beta+\gamma), \quad b = \frac{1}{2}(1-\beta-\gamma), \quad c = 1. \tag{2.10}$$

and one solution to (2.8) is $F(a, b, 1; z)$. Now define

$$F_1(a, b; z) = \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(k!)^2} \sum_{j=0}^{k-1} \left(\frac{1}{a+j} + \frac{1}{b+j} - \frac{2}{1+j} \right) z^k. \tag{2.11}$$

Then a second, linearly independent, solution to (2.8) is

$$F(a, b, 1; z) \log z + F_1(a, b; z), \tag{2.12}$$

as can again be verified by direct computation (see [Car60], §388).

It follows that the function f mapping the upper half plane to the interior of the hyperbolic triangle is

$$f(z) = A \left(\log z + \frac{F_1(a, b; z)}{F(a, b, 1; z)} \right) + B \tag{2.13}$$

where A and B are constants to be determined.

2.2 Hecke Groups

We now wish specialize the results of the previous section to specific triangles we are interested in. Let $m \geq 3$ be either an integer or ∞. Let our hyperbolic triangles have vertices at ρ, i, and ∞, where $\rho = -\exp(-\pi i/m)$; the angles at these vertices are π/m, $\pi/2$ and 0 respectively. Let our function $\Phi_m(w)$ map the upper half plane to this triangle so that 0 maps to ρ, 1 maps to i, and ∞ maps to ∞. The results above assume that the angle is 0 at the vertex to which 0 maps, so we must first use the map $w \to 1/w$ and then map 0 to ∞, 1 to i and ∞ to ρ. Following (2.10), let

$$\alpha = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{m}\right), \quad \beta = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{m}\right). \tag{2.14}$$

Using results described in the previous section, as well as calculations of constants in [Car60], §394, Raleigh ([Ral63], p. 108) determines this function explicitly to be

$$\Phi_m(w) = -\log w + \frac{F_1(\alpha, \beta; 1/w)}{F(\alpha, \beta, 1; 1/w)} + \log A_m \tag{2.15}$$

where

$$\log A_m = -2\psi(1) + \psi(1-\alpha) + \psi(1-\beta) - \pi \sec(\pi/m) \tag{2.16}$$

is a constant. Here $\psi(z) = \Gamma'(z)/\Gamma(z)$.

We are interested in the inverse of $\Phi_m(w)$, which we will call $J_m(z)$, where $2\pi i z/\lambda_m = \Phi_m(w)$. This function satisfies

$$J_m(\rho) = 0, \quad J_m(i) = 1, \quad J_m(i\infty) = \infty,$$

where $\rho = -\exp(-\pi i/m)$, and J_m maps the interior of the hyperbolic triangle with vertices $(\rho, i, i\infty)$ onto $\mathcal{H}$. The angles at the three vertices are $\frac{\pi}{m}, \frac{\pi}{2}$ and 0 respectively. The boundary of the hyperbolic triangle is mapped to $\mathbb{R} \cup \infty$.

Given an integer $m \geq 3$, let G_m be the group of fractional linear transformations, acting discontinuously on the upper half plane $\mathcal{H}$, generated by

$$S(z) = -1/z, \quad T_m(z) = z + \lambda_m, \quad \text{where } \lambda_m = 2\cos(\pi/m).$$

In terms of matrices, these elements are

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T_m = \begin{pmatrix} 1 & \lambda_m \\ 0 & 1 \end{pmatrix}$$

and G_m is a subgroup of $PSL_2(\mathbb{R})$. The groups G_m were first studied by Hecke ([Hec36],[Hec38]) and are referred to as Hecke groups. The case G_3 is just the modular group $\Gamma(1) = PSL_2(\mathbb{Z})$.

The interior of a fundamental region of G_m is the set $D = \{z \in \mathbb{H} : -\lambda/2 < \text{Im } z < \lambda/2, |z| > 1\}$, where $\lambda = \lambda_m$ (see for example [Kat92], p. 101). Since J_m is real valued on the boundary of its hyperbolic triangle, using Schwarz reflection across the imaginary axis we can extend its domain to D; it now maps D and its boundary (except $i\infty$) holomorphically to the entire complex plane. Note that a single reflection has determinant -1 and is not a member of $PSL_2(\mathbb{R})$, but any pair of adjacent reflections will have determinant $+1$ and be a member of G_m. Furthermore by the definition of Schwarz reflection the value of J_m after two adjacent reflections is unchanged. The images of the fundamental region tile the entire upper half plane, and it follows that $J_m(z)$ is automorphic for G_m if we can show that it is meromorphic at infinity.

Since $J_m(z + \lambda_m) = J_m(z)$, the function has a Fourier expansion at $z = i\infty$

$$J_m(z) = \sum_{n=-\infty}^{\infty} a_n q^n$$

where $q = e^{2\pi i z/\lambda_m}$ and $\lambda_m = 2\cos\frac{\pi}{m}$. The function is meromorphic at infinity if all but finitely many $a_n = 0$ for $n < 0$.

In fact it turns out that the function is of the form

$$J_m(z) = \sum_{n=-1}^{\infty} a_n q^n \tag{2.17}$$

as we will show in the next chapter by inverting Φ_m.

Raleigh proves that $a_{-1} = A_m$, where A_m is defined in (2.16); we will reprove this in the next chapter. He shows that $a_{-1} \in \mathbb{Q}$ for $m = 3, 4, 6$ and ∞; Wolfart ([Wol81]) proves that a_{-1} is transcendental for all other values of m.

Raleigh further proves (see also [Aki92]) that for $n \geq 0$ each a_n of the Fourier expansion (2.17) is of the form

$$a_n = \frac{P_n(m^2)}{Q_n a_{-1}^n m^{2n+2}}, \tag{2.18}$$

where P_n is an integer polynomial of degree $n + 1$ and Q_n is an integer. Lehner ([Leh54]), prior to Raleigh's work, proves a similar result using a different normalization of the triangle function.

Akiyama ([Aki92]) proves that the primes dividing Q_n are all less than or equal to $n + 1$. His proof follows Raleigh's use of the Schwarzian derivative. In the next chapter we present a proof of the same result following Lehner's algorithmic method. We then look at which primes divide the denominators in more detail.

CHAPTER 3

Fourier Coefficients of Triangle Functions

This is the primary chapter of the thesis. In this chapter we first prove some general properties of power series and series related to hypergeometric functions in particular. We use these properties to reprove some known results about fourier series of triangle functions. We then describe computer experiments which suggest there are further properties of the denominators of the coefficients with regard to divisibility by primes, and state a detailed conjecture. We show how the conjecture relates to the splitting of primes in the natural ring of integers in which the entries of the relevant Hecke group lie. Finally we use methods of Dwork to prove a special case of the conjecture.

For reference here are the results of the previous chapter which will be most important in this chapter. Given $m \geq 3$, let J_m be the conformal map from the interior of the hyperbolic triangle with vertices $(\rho, i, i\infty)$ onto $\mathcal{H}$. The function satisfies

$$J_m(\rho) = 0, \quad J_m(i) = 1, \quad J_m(i\infty) = \infty,$$

where $\rho = -\exp(-\pi i/m)$, and the angles at the three vertices are $\frac{\pi}{m}, \frac{\pi}{2}$ and 0 respectively. $J_m(z)$ has a Fourier expansion at $z = i\infty$

$$J_m(z) = \sum_{n=-1}^{\infty} a_n q^n \tag{3.1}$$

where $q = e^{2\pi i z/\lambda_m}$ and $\lambda_m = 2\cos\frac{\pi}{m}$. (Note that we will actually prove that the sum starts at -1 in Theorem 9 below.)

The inverse of $J_m(z)$ is $\Phi_m(w)$, where $2\pi iz/\lambda_m = \Phi_m(w)$ and

$$\Phi_m(w) = -\log w + \frac{F_1(\alpha, \beta; 1/w)}{F(\alpha, \beta, 1; 1/w)} + \log A_m \tag{3.2}$$

Here we have

$$\alpha = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{m}\right), \quad \beta = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{m}\right). \tag{3.3}$$

and

$$F(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} z^k, \tag{3.4}$$

where $(a)_k$ is the rising factorial with k terms:

$$(a)_k = a(a+1)\cdots(a+k-1).$$

Also

$$F_1(a, b; z) = \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(k!)^2} \sum_{j=0}^{k-1} \left(\frac{1}{a+j} + \frac{1}{b+j} - \frac{2}{1+j}\right) z^k. \tag{3.5}$$

The function J_m is automorphic for the Hecke group G_m, which is generated by by the transformations

$$S(z) = -1/z, \quad T_m(z) = z + \lambda_m, \quad \text{where } \lambda_m = 2\cos(\pi/m).$$

3.1 Power Series

In this section we present basic results concerning both power series in general and related to the hypergeometric function and J_m in particular. These will be the fundamental tools for proving properties of the coefficients of J_m. We conclude by reproving, using slightly different methods, the results of Lehner, Raleigh and Akiyama mentioned at the end of the last chapter.

Definition 2. Let $v_p(n)$ be the additive p-adic valuation of n, in other words the unique nonnegative integer such that $p^{v_p(n)} \| n$.

Theorem 3 (Gauss). *Let p be prime and $n = \sum_{k=0}^{r} a_k p^k \in \mathbb{Z}^+$, where $a_k \in \mathbb{Z} \cap [0, p)$ for all k. Let $S(n) = \sum_{k=0}^{r} a_k$. Then*

$$v_p(n!) = \frac{n - S(n)}{p - 1}. \tag{3.6}$$

Proof. Exactly $\left\lfloor \frac{n}{p} \right\rfloor$ factors of $n!$ are divisible by p, $\left\lfloor \frac{n}{p^2} \right\rfloor$ factors of $n!$ are divisible by p^2 and so forth. Therefore $v_p(n) = \sum_{k=1}^{r} \left\lfloor \frac{n}{p^k} \right\rfloor$. Now

$$\left\lfloor \frac{n}{p^k} \right\rfloor = a_k + a_{k+1} p + \cdots + a_r p^{r-k},$$

and so

$$(p-1) \left\lfloor \frac{n}{p^k} \right\rfloor = -a_k + (a_k - a_{k+1})p + \cdots + (a_{r-1} - a_r)p^{r-k} + a_r p^{r-k+1}.$$

Summing these together for $k = 1$ to r gives $n - S(n)$. □

An immediate corollary of this theorem is

$$v_p\left(\binom{n+k}{n}\right) = \frac{S(n) + S(k) - S(n+k)}{p-1}. \tag{3.7}$$

Definition 4. Let $\mathbb{Z}(n)[x_0, \ldots, x_k]$ be the set of all polynomials in the variables $x_0, \ldots, x_k$ with coefficients in $\mathbb{Z}$ whose monomial terms $c x_0^{r_0} \cdots x_k^{r_k}$ $(c \in \mathbb{Z})$ all satisfy $\sum_{i=0}^{k} i r_i \leq n$. Similarly let $\mathbb{Z}(n)[x_0, \ldots, x_k, y_0, \ldots, y_m]$ be the set of all polynomials in the variables $x_0, \ldots, x_k, y_0, \ldots, y_m$ with coefficients in $\mathbb{Z}$ whose monomial terms $c x_0^{r_0} \cdots x_k^{r_k} y_0^{a_0} \cdots y_m^{s_m}$ all satisfy $\sum_{i=0}^{k} i r_i + \sum_{i=0}^{m} i s_i \leq n$.

Lemma 5. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be power series expansions about 0 with complex coefficients. Let $f_n(z) = \sum_{k=0}^{n} a_k z^k$. Then the following hold:*

1. *$f_n(z) + g_n(z) = (f+g)_n(z)$, $f_n(z) - g_n(z) = (f-g)_n(z)$, and $f_n(z)g_n(z) = (fg)_n(z)$. Furthermore if c_n is the coefficient of z^n in the expansion of $(f+g)$, $(f-g)$ or (fg), then $c_n \in \mathbb{Z}(n)[a_0, \ldots, a_n, b_0, \ldots, b_n]$.*

2. *If $f(0) = 1$ (i.e., $a_0 = 1$) then $(1/f)_n(z) = 1/f_n(z)$, and if $(1/f)(z) = \sum_{k=0}^{\infty} c_k z^k$ then $c_0 = 1$ and $c_n \in \mathbb{Z}(n)[a_0, \ldots, a_n]$.*

3. *Let $\exp(f)(z) := \sum_{k=0}^{\infty} \frac{f(z)^k}{k!}$ and say that $\exp(f)(z) = \sum_{k=0}^{\infty} c_k z^k$. If $f(0) = 0$ then $\exp(f_n)(z) = \exp(f)_n(z)$, and $c_n \in (1/n!)\mathbb{Z}(n)[a_1, \ldots, a_n]$.*

4. *Say that $f(0) = 0$ and that $g = f^{-1}$. Then $g_n = (f_n)^{-1}$. If $a_1 = 1$ then $b_1 = 1$, and letting $A_k = a_{k+1}$ and $B_k = b_{k+1}$ for all $k \geq 1$, it follows $B_n \in \mathbb{Z}(n)[A_1, \ldots, A_n]$.*

Proof.

1. These follow directly from the definitions. Note that
$fg(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) z^k$.

2. Let $h(z) = -\sum_{k=1}^{\infty} a_k z^k$. Then $1/f(z) = 1/(1-h(z)) = 1 + h(z) + (h(z))^2 + \cdots$.

3. Note that without the condition $f(0) = 0$ is not true in general that $\exp(f_n) = \exp(f)_n$; for example let $f(z) \equiv 1$; then $\exp(f_n) = e$ for all n, whereas $\exp(f)_n = \sum_{k=0}^{n} 1/k!$. However if $f(0) = 0$, so that $f(z) = \sum_{k=1}^{\infty} a_k z^k$, then $f(z)^n$ consists of terms all of order n or greater, so we do have $\exp(f_n) = \exp(f)_n$.

4. Since $f(0) = 0$, we have $f(z) = \sum_{k=1}^{\infty} a_k z^k$ and $g(z) = \sum_{k=1}^{\infty} b_k z^k$. We can determine the coefficients of g given those of f as follows.

Since $z = f(g(z))$, plugging in we have

$$z = \sum_{j=1}^{\infty} a_j \left(\sum_{k=1}^{\infty} b_k z^k \right)^j = \sum_{j=1}^{\infty} a_j z^j \left(\sum_{k=1}^{\infty} b_k z^{k-1} \right)^j .$$

It follows

$$1 = \sum_{j=0}^{\infty} a_{j+1} z^j \left(\sum_{k=0}^{\infty} b_{k+1} z^k \right)^{j+1} .$$

Expanding this out so it is easier to read, we see that

$$\begin{aligned} 1 =& a_1(b_1 + b_2 z + b_3 z^2 + \cdots) + a_2 z(b_1 + b_2 z + b_3 z^2 + \cdots)^2 \\ &+ a_3 z^2 (b_1 + b_2 z + b_3 z^2 + \cdots)^3 + \cdots . \end{aligned}$$

This shows that we can solve for the b_i recursively, the first three steps being

$$\begin{aligned} 1 &= a_1 b_1 \\ 0 &= a_1 b_2 + a_2 b_1^2 \\ 0 &= a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3. \end{aligned}$$

Note that the first equation in which b_k appears consists only of terms involving a_j for $1 \leq j \leq k$ and save for the term $a_1 b_k$ the other terms involve only b_j for $1 \leq j < k$. Therefore knowing $f_n(z)$ suffices to compute $g_n(z)$. Note that for f to have an inverse in a neighborhood of 0 it must be the case that $a_1 \neq 0$.

In the case $a_1 = 1$, it is immediate that $b_1 = 1$, and we have

$$\begin{aligned} 0 =& (B_1 z + B_2 z^2 + \cdots) + A_1 z(1 + B_1 z + B_2 z^2 + \cdots)^2 \\ &+ A_2 z^2 (1 + B_1 z + B_2 z^2 + \cdots)^3 + \cdots . \end{aligned}$$

Again we solve for the B_i recursively, and since the power of z matches the exponent for every A_i and B_i it is clear that $B_n \in \mathbb{Z}(n)[A_1, \ldots, A_n]$ holds.

□

For positive integers m, n, define

$$M_{m,n} = 2^{6n} m^{2n} \prod_{\substack{p \text{ odd prime} \\ p|m,\, p \le n}} p^n \tag{3.8}$$

Lemma 6. *Given m and α, β as in* (3.3)*, let $F(\alpha, \beta, 1; z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Then $c_n M_{m,n} \in \mathbb{Z}$, where $M_{m,n}$ is defined as in* (3.8)*.*

Proof. The techniques used in this proof can be found in Part VIII, Chapter 3, Sections 1 and 2 of [PS98]. From (3.4) and (3.3) we have

$$c_n = \frac{\prod_{k=0}^{n-1}(m - 2 + 4mk)}{(4m)^n n!} \frac{\prod_{k=0}^{n-1}(m + 2 + 4mk)}{(4m)^n n!}.$$

Let $n! = dN$ where d is a product of powers of primes dividing $4m$ and $(N, 4m) = 1$. Let $r \in \mathbb{Z}^+$ be such that $(4m)r \equiv 1 \pmod{N}$. Then $r^n \prod_{k=0}^{n-1}(m-2+4km) \equiv \prod_{k=0}^{n-1}(r(m-2)+k) \pmod{N}$. Now $\prod_{k=0}^{n-1}(r(m-2)+k)$ is divisible by $n!$ (since $\binom{r(m-2)+n-1}{n} \in \mathbb{Z}$) and thus by N; since $(N, r) = 1$ it follows $\prod_{k=0}^{n-1}(m-2+4mk)$ is divisible by N. The same argument shows $\prod_{k=0}^{n-1}(m+2+4mk)$ is divisible by N.

For p prime, $v_p(d) < \frac{n}{p-1}$ by (3.6). If p is odd then at most p^n divides d^2; if $p > n$ then $p \nmid d$. For $p = 2$ at most 2^{2n} divides d^2. □

Theorem 7 (Duke)**.** *Given m and α, β as in* (3.3)*, let*

$$\frac{F(\alpha, \beta, 1; z) \log z + F_1(\alpha, \beta; z)}{F(\alpha, \beta, 1; z)} = \log z + \sum_{n=1}^{\infty} d_n z^n. \tag{3.9}$$

Then $d_n n M_{m,n} \in \mathbb{Z}$, where $M_{m,n}$ is defined as in (3.8)*.*

Proof. In general if y_1, y_2 are solutions of $y'' + p(z)y' + q(z)y = 0$, then the Wronskian $W = y_1 y_2' - y_1' y_2$ satisfies $W' = y_1 y_2'' - y_1'' y_2 = -p(z)W$, and so

$W(z) = Ke^{-\int p(z)\,dz}$ for some constant K. In the case of the hypergeometric differential equation $z(1-z)y'' + [\gamma - (\alpha+\beta+1)z]y' - \alpha\beta y = 0$, we have $W(z) = Kz^{-\gamma}(1-z)^{\gamma-1-\alpha-\beta}$.

In our case the two solutions are $F(\alpha, \beta, 1; z)$ and $F(\alpha, \beta, 1; z)\log z + F_1(\alpha, \beta; z)$, where $\gamma = 1$ and α, β are as in (3.3). We then have

$$W(z) = Kz^{-1}(1-z)^{-1/2} = \frac{K}{z}\sum_{n=0}^{\infty}\frac{(1/2)_n}{n!}z^n.$$

By evaluating zW(z) at 0 it follows $K = 1$. Now $(1/2)_n = \frac{\prod_{j=0}^{n-1}(2j+1)}{2^n}$ and $\prod_{j=0}^{n-1}(2j+1) = \frac{(2n)!}{n!}2^n$. It follows

$$W(z) = \frac{1}{z}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{2^{2n}}z^n. \tag{3.10}$$

Let $F(z) = F(\alpha, \beta, 1; z)$ and $G(z) = F(\alpha, \beta, 1; z)\log z + F_1(\alpha, \beta; z)$. Set $w = \frac{G(z)}{F(z)}$ and assume z is a function of w. Then taking the derivative with respect to w of both sides we get $1 = \frac{(FG' - F'G)(z)}{F(z)^2}\frac{dz}{dw} = W(z)F(z)^{-2}\frac{dz}{dw}$. By Lemma 6, if $F(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$, then $a_n M_{m,n} \in \mathbb{Z}$, and by Lemma 5 if $F(z)^{-2} = 1 + \sum_{n=1}^{\infty} b_n z^n$ then $b_n M_{m,n} \in \mathbb{Z}$. By (3.10), if $zW(z) = \sum_{n=0}^{\infty} c_n z^n$ then $c_n M_{m,n} \in \mathbb{Z}$. It follows $\frac{dw}{dz} = W(z)F(z)^{-2} = \frac{1}{z} + \sum_{n=0}^{\infty} d_{n+1}z^n$ where $d_n M_{m,n} \in \mathbb{Z}$. Now integrate with respect to z to get $w = \log z + \sum_{n=1}^{\infty}\frac{d_n}{n}z^n$ where the integration constant is 0 by inspection. The result follows. □

Corollary 8. *Given α, β as in* (3.3)*, let*

$$z\exp\left(\frac{F_1(\alpha, \beta; z)}{F(\alpha, \beta, 1; z)}\right) = z + \sum_{n=2}^{\infty} c_{n-1}z^n. \tag{3.11}$$

Then $c_n L_n M_{m,n} \in \mathbb{Z}$ where $M_{m,n}$ is defined as in (3.8) *and L_n is a number all of whose prime divisors are less than or equal to n.*

Proof. Follows immediately from Theorem 7 and Lemma 5 by exponentiating both sides of (3.9). □

We can now describe the coefficients of J_m in some detail.

Theorem 9. *$J_m(z) = \sum_{n=-1}^{\infty} a_n q^n$ where*

$$a_n(a_{-1}^n L_{n+1} M_{m,n+1}) \in \mathbb{Z} \text{ for } n \geq 0. \tag{3.12}$$

where $M_{m,n}$ is defined as in (3.8) and L_n is a number all of whose prime divisors are less than or equal to n.

In addition, if, for all $n \geq 1$, the coefficients c_n of (3.11) are in $\mathbb{Z}_p$ for some prime p, then so are $a_n a_{-1}^n$ (for all $n \geq -1$).

Proof. In (3.2), substitute w for $1/w$ (this converts the power series to be around 0 rather than ∞) and let $F_1 = F_1(\alpha, \beta; w)$ and $F = F(\alpha, \beta, 1; w)$. Exponentiating both sides results in

$$q = A_m w \exp(F_1/F), \tag{3.13}$$

where $q = e^{2\pi i z/\lambda_m}$. Inverting this will result in $w = 1/J_m(z) = 1/\left(\sum_{n=-\infty}^{\infty} a_n q^n\right)$. Calculating as in Lemma 5 results in $1/w = J_m(z) = \sum_{n=-1}^{\infty} a_n q^n$ with $a_{-1} = A_m$. Now let $u = q/a_{-1}$. We have

$$\begin{aligned} u &= w \exp(F_1/F) && = w + c_1 w^2 + c_2 w^3 + \cdots \\ w &= u\left(1 + \sum_{n=1}^{\infty}(a_{n-1}a_{-1}^{n-1})u^n\right)^{-1} && = u + d_1 u^2 + d_2 u^3 + \cdots \end{aligned}$$

The latter implies $1 + \sum_{n=1}^{\infty}(a_{n-1}a_{-1}^{n-1})u^n = (1 + \sum_{n=1}^{\infty} d_n u^n)^{-1}$. From Lemma 5 we thus see that $d_n \in \mathbb{Z}(n)[c_1, \ldots, c_n]$ and $a_n a_{-1}^n \in \mathbb{Z}(n+1)[d_1, \ldots, d_{n+1}]$. From Corollary 8 we have $c_n \in \frac{\mathbb{Z}}{L_n M_{m,n}}$. Putting this together gives the first result, and it is also clear that if the c_n are in $\mathbb{Z}_p$ then so are $a_n a_{-1}^n$. □

This in turn gives us a different proof of Akiyama's theorem.

Corollary 10. *Let* $J_m(z) = \sum_{n=-1}^{\infty} a_n q^n$ *where*

$$a_n = \frac{C_n}{a_{-1}^n D_n m^{2n+2}}$$

where $C_n, D_n \in \mathbb{Z}$, $(C_n, D_n) = 1$. *Then for* $n \geq 1$ *all primes dividing* D_n *are less than or equal to* $n + 1$.

Proof. Follows immediately from Theorem 9. □

3.2 Computer Experiments

Computer experiments were performed to explore properties of the rational integer part of the fourier coefficients of the J_m. In this section we present the Mathematica code and some of the experimental results. Mathematica version 5.0 ([Wol03]) was used, and functions undocumented here are internal Mathematica functions. More information on them can be found in the Mathematica documentation. A Mathematica notebook containing code for the experiments is available at [Leo08].

We follow Lehner in that we construct the inverse function Φ_m of J_m and then invert it, as in Theorem 9. We first define the functions F and F_1. In the following code n is the number of terms to retain in the series. Note also that z is used as the Mathematica variable for all power series regardless of whether the actual variable in this thesis is w, z or $q = e^{2\pi i z/\lambda_m}$. It will be clear from the context which is meant.

F[n_, m_] :=

Module[{α, β},

$\alpha = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{m}\right)$;

$\beta = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{m}\right)$;

Series[Hypergeometric2F1[α, β, 1, z], {z, 0, n}]

]

F1[n_, m_] :=

Module[{α, β},

$\alpha = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{m}\right)$;

$\beta = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{m}\right)$;

$$\texttt{Series}\Big[\sum_{k=1}^{n}\left(\frac{\texttt{Pochhammer}[\alpha,k]\,\texttt{Pochhammer}[\beta,k]}{(k!)^2}\left(\sum_{j=0}^{k-1}\left(\frac{1}{\alpha+j}+\frac{1}{\beta+j}-\frac{2}{j+1}\right)\right)\right)z^k,\ \{z,0,n\}\Big]$$

]

The following are two functions to compute the series $qJ_m(A_m q)$ from its inverse Φ_m. The first uses the direct method of Theorem 9. The second uses the Wronskian of Theorem 7. The second is a little faster (it does not use F_1) although not significantly, since most of the time is spent in exponentiation and inversion of the series. Both factor out A_m immediately so all computation is done purely in rational integers.

ComputeInverseFunction[terms_, m_] :=

$z/($InverseSeries[z exp[F1[terms, m]/F[terms, m]]])

```
ComputeInverseFunction2[terms_, m_] :=
  Module[{W},
    W = Series[z^-1 (1 - z)^(-1/2), {z, 0, terms}];
    z/(InverseSeries[exp [Integrate[W/(F[terms, m]^2), z]]])
  ]
```

The following code is used to format and print the power series so that its coefficients are in factored form. The function `IntegerFactorForm` factors an individual integer; for example 24 is printed as $2^3 3^1$. Note that `FactorComplete` is typically set to `False` as we will not care about factoring the rather large integers which can appear in the numerators, and which can take considerable time to fully factor. It can be set to `True` if needed. The function `FactorForm` then prints a rational number in factored form; for example 24/35 would be printed as $\frac{2^3 3^1}{5^1 7^1}$. The final function `FactorPoly` reconstructs an entire polynomial with rational coefficients in factored form.

```
IntegerFactorForm[n_] :=
  Module[{f},
    If[n == 1, RowBox[{"1"}],
      f = Apply[ToString, FactorInteger[n, FactorComplete → False], {2}];
      RowBox[Apply[SuperscriptBox, f, {1}]]]
  ]
```

```
FactorForm[n_] :=
  Module[{f1, f2},
    If[IntegerQ[n], IntegerFactorForm[n],
      f1 = IntegerFactorForm[Numerator[n]];
      f2 = IntegerFactorForm[Denominator[n]];
      FractionBox[f1, f2]]
  ]
FactorPoly[poly_, x_] :=
  Module[{i, outp, coeff},
    Off[FactorInteger :: facnf];
    outp = {FactorForm[Coefficient[poly, x, 0]]};
    For[i = 1, i < Length[poly], i++,
      coeff = Coefficient[poly, x, i];
      If[coeff < 0,
        outp = Append[outp, "-"]; coeff = Abs[coeff],
        outp = Append[outp, "+"]];
      outp = Append[outp, FactorForm[coeff]];
      outp = Append[outp, SuperscriptBox[ToString[x], ToString[i]]]
    ];
    On[FactorInteger :: facnf];
    RowBox[outp] //DisplayForm
  ]
```

We next present some utility functions. The first replaces a polynomial with rational integer coefficients with a similar polynomial in which only the denominators have been retained and the numerators replaced by 1. This is useful because we will be concerned with looking only at denominators of coefficients.

```
DenominatorsOnly[poly_, z_] :=
  Module[{ot, i, mx}, ot = 0;
    mx = Exponent[poly, z];
    For[i = 0, i ≤ mx, i++,
      ot += 1/Denominator[Coefficient[poly, z, i]] z^i;
    ];
    ot]
```

The next utilities check to see that no denominators of a power series are divisible by primes congruent to 0 or ± 1 mod m; a counterexample would invalidate Conjecture 11 below.

```
CheckPrime[p_, m_] :=
  Not[Mod[p, m] == 0 || Mod[p, m] == 1 || Mod[p, m] == m - 1]
```

```
CheckPrimes[n_, m_] :=
  Module[{cp}, cp[k_] := CheckPrime[k[[1]], m];
    If[n == 1, True, Apply[And, Map[cp, FactorInteger[n]]]]]
```

```
CheckDenominators[poly_, z_, m_] :=
  Module[{ot, i, mx}, ot = True;
    mx = Exponent[poly, z];
    For[i = 0, i ≤ mx, i++,
      ot = ot && CheckPrimes[Denominator[Coefficient[poly, z, i]], m];
    ];
    ot]
```

Finally here is an example of the computation of the first 50 nonconstant terms of $qJ_m(A_m 2^6 m^3 q)$ for $m = 5$. Note that the number of terms and m can both be easily changed. The output shows the factorization of denominators of the power series.

```
Module[{terms, m, K, KN},
  terms = 50; m = 5;
  K = ComputeInverseFunction2[terms, m];
  KN = Normal[K] /. z → (2^6 m^3) z;
  FactorPoly[DenominatorsOnly[KN, z], z]
]
```

$$
\begin{aligned}
&1 + 1z^{1} + 1z^{2} + 1z^{3} + 1z^{4} + 1z^{5} + 1z^{6} + \frac{1}{7^{1}}z^{7} + 1z^{8} + \frac{1}{3^{1}7^{1}}z^{9} + 1z^{10} + \frac{1}{3^{1}}z^{11} \\
&+ 1z^{12} + \frac{1}{13^{1}}z^{13} + \frac{1}{7^{2}}z^{14} + \frac{1}{13^{1}}z^{15} + \frac{1}{7^{2}13^{1}}z^{16} + \frac{1}{7^{1}13^{1}17^{1}}z^{17} + \frac{1}{3^{2}7^{1}13^{1}}z^{18} \\
&+ \frac{1}{7^{1}13^{1}17^{1}}z^{19} + \frac{1}{3^{2}7^{1}13^{1}17^{1}}z^{20} + \frac{1}{3^{1}7^{3}13^{1}17^{1}}z^{21} + \frac{1}{13^{1}17^{1}}z^{22} \\
&+ \frac{1}{3^{1}7^{3}13^{1}17^{1}23^{1}}z^{23} + \frac{1}{3^{1}7^{2}13^{1}17^{1}}z^{24} + \frac{1}{7^{2}13^{1}17^{1}23^{1}}z^{25} + \frac{1}{3^{1}7^{2}13^{2}17^{1}23^{1}}z^{26} \\
&+ \frac{1}{3^{4}7^{2}17^{1}23^{1}}z^{27} + \frac{1}{7^{4}13^{2}17^{1}23^{1}}z^{28} + \frac{1}{3^{4}13^{2}17^{1}23^{1}}z^{29} + \frac{1}{3^{3}7^{4}13^{2}17^{1}23^{1}}z^{30} \\
&+ \frac{1}{7^{3}13^{2}17^{1}23^{1}}z^{31} + \frac{1}{3^{3}7^{3}13^{2}17^{1}23^{1}}z^{32} + \frac{1}{3^{3}7^{3}13^{2}17^{1}23^{1}}z^{33} + \frac{1}{7^{3}13^{2}17^{2}23^{1}}z^{34} \\
&+ \frac{1}{3^{3}7^{5}13^{2}23^{1}}z^{35} + \frac{1}{3^{5}13^{2}17^{2}23^{1}}z^{36} + \frac{1}{7^{5}13^{2}17^{2}23^{1}37^{1}}z^{37} \\
&+ \frac{1}{3^{5}7^{4}13^{2}17^{2}23^{1}}z^{38} + \frac{1}{3^{4}7^{4}13^{3}17^{2}23^{1}37^{1}}z^{39} + \frac{1}{7^{4}17^{2}23^{1}37^{1}}z^{40} \\
&+ \frac{1}{3^{4}7^{4}13^{3}17^{2}23^{1}37^{1}}z^{41} + \frac{1}{3^{4}7^{6}13^{3}17^{2}23^{1}37^{1}}z^{42} + \frac{1}{13^{3}17^{2}23^{1}37^{1}43^{1}}z^{43} \\
&+ \frac{1}{3^{4}7^{6}13^{3}17^{2}23^{1}37^{1}}z^{44} + \frac{1}{3^{6}7^{5}13^{3}17^{2}23^{1}37^{1}43^{1}}z^{45} + \frac{1}{7^{5}13^{3}17^{2}23^{2}37^{1}43^{1}}z^{46} \\
&+ \frac{1}{3^{6}7^{5}13^{3}17^{2}37^{1}43^{1}47^{1}}z^{47} + \frac{1}{3^{5}7^{5}13^{3}17^{2}23^{2}37^{1}43^{1}}z^{48} \\
&+ \frac{1}{7^{8}13^{3}17^{2}23^{2}37^{1}43^{1}47^{1}}z^{49} + \frac{1}{3^{5}13^{3}17^{2}23^{2}37^{1}43^{1}47^{1}}z^{50}
\end{aligned}
$$

One may notice two phenomena here. The primes that appear in the denominators are only those not dividing $2m$ and not congruent to $\pm 1 \bmod m$. Furthermore the prime p first appears at place z^{p^k}, where here $k = 1$ except for $p = 3$ for which $k = 2$. More thorough experiments for other values of m and more terms support these hypotheses.

3.3 Conjecture

Based on the results of the computer experiments described in the previous section, we conjecture the following.

Conjecture 11. *Let $m = 5$ or $m \geq 7$. Let $J_m(z) = \sum_{n=-1}^{\infty} a_n q^n$, and let*

$$a_n = \frac{C_n}{D_n a_{-1}^n 2^{6n+6} m^{3n+3}}$$

where $C_n, D_n \in \mathbb{Z}$, $(C_n, D_n) = 1$. Then the set of primes dividing some element of $\{D_n : n \geq 1\}$ is $\{p : p \nmid 2m \text{ and } p \not\equiv \pm 1 \pmod{m}\}$. Furthermore if n_0 is the least n for which p divides the denominator of D_n, then $n_0 = p^k - 1$ for some $k \geq 1$.

If this conjecture is true it answers in the negative the question by Pisot ([Ral63], p. 111) concerning whether there exist $J_m(z)$ for $m \neq 3, 4, 6, \infty$ such that the denominators of $J_m(z)$ are bounded. Furthermore the conjecture says that (for all $m \geq 3$) if $p \equiv 0 \pmod{m}$ or $p \equiv \pm 1 \pmod{m}$, then $\frac{C_n}{D_n} \in \mathbb{Z}_p$ for all n.

3.4 Cyclotomic Fields

In this section we discuss how the conjecture relates to the splitting of primes in cyclotomic fields.

Let $\zeta_n = e^{2\pi i/n}$ be a primitive nth root of unity. Then $\mathbb{Q}(\zeta_n)$ is a cyclotomic field whose ring of integers is $\mathbb{Z}[\zeta_n]$. The cyclotomic field $\mathbb{Q}(\zeta_n)$ has a unique maximal real subfield $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$, and $[Q(\zeta_n) : \mathbb{Q}(\zeta_n + \zeta_n^{-1})] = 2$. Let $n = 2m$; then $\zeta_n + \zeta_n^{-1} = 2\cos(\pi/m) = \lambda_m$. The ring of integers of $\mathbb{Q}(\lambda_m)$ is $\mathbb{Z}[\lambda_m]$ (see [Was97], Proposition 2.16). Note also that the elements of G_m have entries in $\mathbb{Z}[\lambda_m]$.

We now describe how primes split in cyclotomic fields and their maximal real subfields (see [Mar77], pages 108–110 and 118, and [Was97], Theorem 2.13).

Theorem 12. *Fix n and let $L = \mathbb{Q}(\zeta_n)$ and $K = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$. Let $p \in \mathbb{Q}$ be a prime not dividing n. Then p splits into $g = \phi(n)/f$ distinct primes in L, where f is the the smallest positive integer such that $p^f \equiv 1 \pmod{n}$. Also p splits into $g = \phi(n)/2f$ distinct primes in K, where f is the the smallest positive integer such that $p^f \equiv \pm 1 \pmod{n}$. In either case each prime has residue class f. In particular, p splits completely in L iff $p \equiv 1 \pmod{n}$ and splits completely in K iff $p \equiv \pm 1 \pmod{n}$.*

Proof. The Galois group of the cyclotomic field, $G = \mathrm{Gal}(L/\mathbb{Q})$, is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^*$. Since $p \nmid n$ it is unramified and splits into g distinct primes each with inertial index f, where $fg = |G| = \phi(n)$. Let P be a prime over p and $\bar{G}$ be the Galois group of O_L/P over $\mathbb{Z}/p$; by definition $f = \left|\bar{G}\right|$. Since $\bar{G}$ is the Galois group of an extension of finite fields, it is cyclic and generated by the unique frobenius element which sends every $\bar{k} \in \mathbb{Z}/p$ to $\bar{k}^p$. There is an isomorphism from the decomposition group, $D = \{\tau \in G : \tau\mathbb{Q} = \mathbb{Q}\}$, to $\bar{G}$. The member of D corresponding to the frobenius element is the frobenius automorphism σ; it satisfies

$$\sigma(x) \equiv x^p \bmod P \qquad \text{for all } x \in O_L$$

and is the unique member of G with this property. Since $L = \mathbb{Q}(\zeta_n)$ we have in fact $\sigma(\zeta_n) = \zeta_n^p$, and $\sigma^f = 1_G$ iff $\zeta_n^{p^f} = \zeta_n$ iff $p^f \equiv 1 \pmod{n}$, which proves the first part of the theorem.

If we restrict σ to K then it must be the frobenius automorphism of $G' = \mathrm{Gal}(K/\mathbb{Q})$, since it satisfies $\sigma|_K(x) \equiv x^p \bmod Q$ for all $x \in O_K$ where Q is any prime of O_K lying above p and below P. We have $\sigma|_K^{f'} = 1_{G'}$ iff $\zeta_n^{p^{f'}} = \zeta_n^{\pm 1}$ iff $p^{f'} \equiv \pm 1 \pmod{n}$, which proves the second part of the theorem. □

If we let $n = 2m$ then this theorem says that a prime $p \nmid 2m$ splits completely in $\mathbb{Z}[\lambda_m]$ iff $p \equiv \pm 1 \pmod{2m}$. Note the similarity to the condition for a prime to not appear in the denominators in Conjecture 11. There the condition is $p \equiv \pm 1 \pmod{m}$, which is equivalent if m is odd but weaker if m is even. As an example, consider the denominators of the coefficients of J_{12}:

$$\begin{aligned}
&1 + 1z^1 + 1z^2 + 1z^3 + 1z^4 + \frac{1}{5^1}z^5 + 1z^6 + \frac{1}{5^1 7^1}z^7 + 1z^8 + \frac{1}{7^1}z^9 \\
&+ \frac{1}{5^2}z^{10} + 1z^{11} + \frac{1}{5^2}z^{12} + \frac{1}{5^1}z^{13} + \frac{1}{5^1 7^2}z^{14} + \frac{1}{5^3}z^{15} + \frac{1}{7^2}z^{16} \\
&+ \frac{1}{5^3 7^1 17^1}z^{17} + \frac{1}{5^2 7^1}z^{18} + \frac{1}{5^2 7^1 17^1 19^1}z^{19} + \frac{1}{5^4 7^1 17^1}z^{20} \\
&+ \frac{1}{7^3 17^1 19^1}z^{21} + \frac{1}{5^4 17^1 19^1}z^{22} + \frac{1}{5^3 7^3 17^1 19^1}z^{23} \\
&+ \frac{1}{5^3 7^2 17^1}z^{24} + \frac{1}{5^6 7^2 17^1 19^1}z^{25} + \frac{1}{7^2 17^1 19^1}z^{26} + \frac{1}{5^6 7^2 17^1 19^1}z^{27} \\
&+ \frac{1}{5^5 7^4 17^1 19^1}z^{28} + \frac{1}{5^5 17^1 19^1 29^1}z^{29} + \frac{1}{5^7 7^4 17^1 19^1}z^{30}
\end{aligned}$$

Here the primes 5 and 7 appear in denominators while 11 and 13 do not, whereas all four split into two primes in $Z[\lambda_{12}]$ with $f = 2$. Likewise 11, 13 and 23 all do not appear in denominators, whereas only 23 splits completely in $Z[\lambda_{12}]$.

In any case it appears that the splitting of primes in $Z[\lambda_m]$ seems to be behind which primes appear in the denominators. Intuitively primes that split more should be canceled by factors in the numerator and not appear in the denominator. However at this point we do not have a precise explanation for this behavior. It does appear clear that $Z[\lambda_m]$ is the proper ring in which to further investigate properties of the behavior of the fourier coefficients.

3.5 Dwork's Method

Dwork's work ([Dwo69, Dwo73]) provides a method for proving integrality results for power series solutions to certain generalized hypergeometric differential equations. This method was extended by Lian and Yau ([LY98]) and Zudilin ([Zud02]). In this section we describe Dwork's method, and in the next section we use it to partially solve Conjecture 11.

Let p be prime and define C_p to be the set of all p-integral rational numbers which are not in $\mathbb{Z}^- \cup \{0\}$. Define $\theta : C_p \to C_p$ so that $\theta(x)$ is the unique number such that $p\theta(x) - x \in \mathbb{Z} \cap [0, p-1]$. If $x = a/b$ where $(a, b) = 1$ and $p \nmid b$, and $\theta(x) = c/d$ where (c, d) and $p \nmid d$ $(a, b, c, d \in \mathbb{Z},\ b, d > 0)$, then we have $\theta(x)p - x = \frac{pcb - ad}{bd}$. For this to be an integer we must have $d|pcb$ which implies $d|b$, and also $b|ad$ which implies $b|d$, so in fact $b = d$. We now have $p\theta(x) - x = \frac{pc-a}{b}$. We need to choose c such that $\frac{pc-a}{b} \in \mathbb{Z} \cap [0, p-1]$. Since $(pac, b) = 1$ we can choose c so that $c \equiv ap^{-1} \pmod{b}$. This means $c = a(p^{-1} + \ell b) + mb$ for some $\ell, m \in \mathbb{Z}$, since both p^{-1} and c are determined only mod b. Let $n = a\ell + m$, so that $c = ap^{-1} + nb$. Plugging in we get $\frac{pc-a}{b} = \frac{p(a(p^{-1}+nb)-a}{b} = \frac{a(1+kb)+pnb-a}{b} = ak + pn$ where k is fixed and depends on the particular choice of p^{-1}. We choose the unique n so that $ak + pn \in [0, p-1]$, and this determines c uniquely. Note that if $x \in \mathbb{Z}^+$ then $\theta(x) \in \mathbb{Z}^+$ as well.

If $a \in (0, b)$ then $c \in (0, b)$ as well, since we require $pc - a \in [0, bp - b]$ which implies $pc \in (0, bp)$. Since $c \equiv ap^{-1} \pmod{b}$ it is determined uniquely if we know p mod b, and given c we can determine a uniquely from the same congruence. We have thus proven the following lemma, which is Lemma 10 of [Zud02].

Lemma 13. *Given $b \geq 2$, $(b, p) = 1$, let $S = \{a/b : a \in (0, b), (a, b) = 1\} \subseteq C_p$. Then $\theta : S \to S$ is a bijection, with $\theta(a/b) = c/b$ where $c \in (0, b)$ and $c \equiv ap^{-1}$*

(mod b).

We can describe the relationship between x and $\theta(x)$ more simply by looking at their p-adic expansions. Since x is p integral, it can be written uniquely as $\sum_{n=0}^{\infty} a_n p^n$, where $a_n \in \mathbb{Z} \cap [0, p-1]$. Then $\theta(x) = 1 + \sum_{n=0}^{\infty} a_{n-1} p^n$, and $p\theta(x) - x = p - a_0$. The significance of $p\theta(x) - x$ is that it is the smallest non-negative integer k such that $x + k$ is divisible by p, and thus corresponds to the first term in the rising factorial of x that will be divisible by p.

We prove a variant of Lemma 6. This is a simplified version of Lemmas 2.1 and 3.2 of [Dwo73].

Lemma 14. *Given $a, b \in C_p$, let $A(k) = \frac{(a)_k (b)_k}{(k!)^2}$. Let $k = k_0 + k_1 p$ where $k_0 \in [0, p)$. Then*

$$v_p(A(k)) \geq \rho(k_0, a)[1 + v_p(k_1 + \theta(a))] + \rho(k_0, b)[1 + v_p(k_1 + \theta(b))] \qquad (3.14)$$

where

$$\rho(k_0, a) = \begin{cases} 1 & \text{if } k_0 > p\theta(a) - a; \\ 0 & \text{otherwise.} \end{cases} \qquad (3.15)$$

In particular, $F(a, b, 1; z) = \sum_{k=0}^{\infty} A(k) z^k \in \mathbb{Z}_p[[z]]$.

Proof. It suffices to prove $v_p((a)_k / k!) \geq \rho(k_0, a)[1 + v_p(k_1 + \theta(a))]$ for all $a \in C_p$ and $k \geq 0$. First note that

$$\frac{(a)_k}{k!} = \frac{(a)_{pk_1}}{(pk_1)!} \cdot \frac{(a + pk_1)_{k_0}}{\prod_{m=1}^{k_0} (pk_1 + m)}.$$

For $j \geq 1$ and $i \geq 0$, the interval $[1 + ip^j, (i+1)p^j]$ is a complete set of representatives mod p^j and exactly one element (namely the last one) is divisible by p^j. Similarly the set $\{a + n : n \in [ip^j, (i+1)p^j - 1]\}$ is a complete set of representatives mod p^j and exactly one element of the set (perhaps earlier than the last element) is divisible by p^j. It follows $v_p((a)_{pk_1}/(pk_1)!) \geq 0$, and so we examine

$$\frac{(a+pk_1)_{k_0}}{\prod_{m=1}^{k_0}(pk_1+m)},$$

which has k_0 terms in each the numerator and denominator. No term in the denominator is divisible by p, and at most one term in the numerator is divisible by p. From the definition of θ we know that $v_p(a+pk_1+n) \geq 1$ iff $n = p\theta(a)-a+\ell p$ for some $\ell \in \mathbb{Z}$. Noting that $p\theta(a)-a \in [0,p)$, we see that such a term $a+pk_1+n$ exists in the numerator iff $p\theta(a)-a \leq k_0-1$ iff $\rho(k_0,a)=1$. Furthermore for this n we have $v_p(a+pk_1+n) = v_p(a+pk_1+p\theta(a)-a) = 1+v_p(k_1+\theta(a))$. $\square$

We now prove a lemma that will allow us to considerably simplify Dwork's proof in the special case we are concerned with. Although some ideas related to this lemma appear in Dwork's work it does not appear to be explicitly stated.

Lemma 15. *Given $a,b \in C_p$, let $A(k) = \frac{(a)_k(b)_k}{(k!)^2}$ and $A'(k) = \frac{(\theta(a))_k(\theta(b))_k}{(k!)^2}$ for $k \geq 0$. Let $k = k_0+pk_1$ where $k \in [0,p)$ and $k_1 \geq 0$. If either $\rho(k_0,a)=1$ or $\rho(k_0,b)=1$ then $A(k) \equiv 0 \pmod{p}$. Otherwise $A(k) = A(k_0)B$ where $B \equiv A'(k_1) \pmod{p}$.*

Proof. If either $\rho(k_0,a)=1$ or $\rho(k_0,b)=1$ then $A(k) \equiv 0 \pmod{p}$ follows immediately from Lemma 14. So assume $\rho(k_0,a)=\rho(k_0,b)=0$. We will separate $A(k)$ into products of k_0 terms and pk_1 terms as was done in the proof of Lemma 14, but this time the k_0 terms will be the first terms rather than the last ones.

We have $A(k) = A(k_0)B$, where

$$B = \frac{(a+k_0)_{pk_1}(b+k_0)_{pk_1}}{[(k_0+1)_{pk_1}]^2}.$$

Consider the set of factors of the numerator $\{a+k_0+j : p(m-1) \leq j < pm\}$ for each $1 \leq m \leq k_1$, and the corresponding set of factors of the denominator

$\{k_0 + 1 + j : p(m-1) \le j < pm)\}$. Each set forms a complete set of representatives mod p, and so all terms not divisible by p cancel each other mod p. The term divisible by p in the denominator lies in the interval $[k_0 + p(m-1)+1, k_0 + pm]$, and so must be pm. The term divisible by p in the numerator lies in the interval $[a+k_0+p(m-1), a+k_0+pm-1]$, and so must be $p\theta(a)+p(m-1)$ if $p\theta(a)-a \ge k_0$ (given $a+k_0+p(m-1)+j$, where $j \in [0,p)$, let $j = p\theta(a)-a-k_0$). The condition $p\theta(a) - a \ge k_0$ is equivalent to $\rho(k_0, a) = 0$, which we have assumed to be true. Taking the product over all $1 \le m \le k_1$, we get $\frac{(\theta(a))_{k_1}}{k_1!}$. After repeating the argument with b, the lemma follows. $\square$

We now prove a special case of Theorem 4.1 of [Dwo73]. We follow Dwork's proof, but simplify details where possible.

Theorem 16 (Dwork). *Let* $F(a, b, c; z)$ *and* $F_1(a, b; z)$ *be defined as in* (3.4) *and* (3.5)*. Let* $a, b \in C_p$*. Then*

$$\frac{F_1(\theta(a), \theta(b); z^p)}{F(\theta(a), \theta(b), 1; z^p)} \equiv p\frac{F_1(a, b; z)}{F(a, b, 1; z)} \pmod{p\mathbb{Z}_p[[z]]}. \tag{3.16}$$

Proof. Let $F(z) = F(a, b, 1; z) = \sum_{k=0}^{\infty} A(k) z^k$ and $F_1(z) = F_1(a, b; z) = \sum_{k=1}^{\infty} A(k)D(k)z^k$ where $A(k) = \frac{(a)_k(b)_k}{(k!)^2}$ and

$$D_x(k) = \sum_{j=0}^{k-1} \frac{1}{x+j}; \tag{3.17}$$

$$D(k) = D_a(k) + D_b(k) - 2D_1(k). \tag{3.18}$$

Let $F'(z) = F(\theta(a), \theta(b), 1; z) = \sum_{k=0}^{\infty} A'(k)z^k$ and $F_1'(z) = F_1(\theta(a), \theta(b); z) = \sum_{k=1}^{\infty} A'(k)D'(k)z^k$ where $A'(k) = \frac{(\theta(a))_k(\theta(b))_k}{(k!)^2}$ and $D'(k) = D_{\theta(a)}(k) + D_{\theta(b)}(k) - 2D_1(k)$. With this notation we wish to prove that

$$\frac{F_1'(z^p)}{F'(z^p)} \equiv p\frac{F_1(z)}{F(z)} \pmod{p\mathbb{Z}_p[[z]]}. \tag{3.19}$$

Note that by Lemma 14 we have $F(z), F'(z) \in \mathbb{Z}_p[[z]]$ and, since both their constant terms are 1, by Lemma 5 we also have $1/F(z), 1/F'(z) \in \mathbb{Z}_p[[z]]$. On the other hand in general $F_1(z), F_1'(z) \in \mathbb{Q}_p[[z]]$.

Let $G_x(z) = \sum_{k=1}^{\infty} A(k)D_x(k)z^k$ and $G_x'(z) = \sum_{k=1}^{\infty} A'(k)D_{\theta(x)}(k)z^k$. Then $F_1(z) = G_a(z)+G_b(z)-2G_1(z)$, $F_1'(z) = G_a'(z)+G_b'(z)-2G_1'(z)$, and by linearity to prove (3.19) it suffices to prove

$$\frac{G_x'(z^p)}{F'(z^p)} \equiv p\frac{G_x(z)}{F(z)} \pmod{p\mathbb{Z}_p[[z]]} \tag{3.20}$$

for all $x \in \{a, b, 1\}$. This is equivalent to

$$F(z)G_x'(z^p) - pF'(z^p)G_x(z) \in p\mathbb{Z}_p[[z]] \tag{3.21}$$

for all $x \in \{a, b, 1\}$ since $F(z), F'(z), 1/F(z), 1/F'(z) \in \mathbb{Z}_p[[z]]$.

Let $F(z)G_x'(z^p) - pF'(z^p)G_x(z) = \sum_{k=0}^{\infty} C_x(k)z^k$. Then proving (3.21) for all $x \in \{a, b, 1\}$ is equivalent to proving

$$C_x(k) \equiv 0 \pmod{p} \quad \text{for all } k \geq 0, x \in \{a, b, 1\}, \tag{3.22}$$

where, letting $k = k_0 + pk_1$ ($k_0 \in [0, p)$), we have by calculation

$$C_x(k_0 + pk_1) = \sum_{j=0}^{k_1} A(k_0 + pj)A'(k_1 - j)[D_{\theta(x)}(k_1 - j) - pD_x(k_0 + pj)]. \tag{3.23}$$

Consider $D_x(k_0 + pj) - D_x(pj) = \sum_{i=0}^{k_0-1} \frac{1}{x+i+pj}$. The only numbers of the form $x+i+pj$ divisible by p are those for which $i = p\theta(x) - x + p\ell$ for some $\ell \in \mathbb{Z}$; for $i \in [0, k_0)$ the only possible value is $i = p\theta(x) - x$ which occurs iff $\rho(k_0, x) = 1$. It follows

$$D_x(k_0 + pj) - D_x(pj) \equiv \frac{1}{p}\frac{\rho(k_0, x)}{j + \theta(x)} \pmod{\mathbb{Z}_p}. \tag{3.24}$$

Similarly the only denominators in the sum for $D_x(pj)$ that are divisible by p are those of the form $x + p\theta(x) - x + p\ell$ for $\ell \in [0, j)$, and it follows

$$D_x(pj) \equiv \frac{1}{p}D_{\theta(x)}(j) \pmod{\mathbb{Z}_p}. \tag{3.25}$$

From (3.24) and (3.25) it follows $pD_x(k_0 + pj) - D_{\theta(x)}(j) \equiv \frac{\rho(k_0,x)}{j+\theta(x)} \pmod{p}$, and then from Lemma 14 it follows

$$A(k_0 + pj)[pD_x(k_0 + pj) - D_{\theta(x)}(j)] \equiv 0 \pmod{p}. \tag{3.26}$$

Note that we have used here the fact that $x \in \{1, a, b\}$. If $\rho(k_0, x) = 0$, then (3.26) is obvious. Note that $\rho(k_0, 1) = 0$ for all $k_0 \in [0, p)$. Otherwise ($x = a$ or $x = b$) we have $v_p(A(k_0 + pj)) \geq 1 + v_p(j + \theta(x))$ by Lemma 14.

From (3.23) and (3.26) it then follows

$$C_x(k_0+pk_1) \equiv \sum_{j=0}^{k_1} A(k_0+pj)A'(k_1-j)[D_{\theta(x)}(k_1-j)-D_{\theta(x)}(j)] \pmod{p}. \tag{3.27}$$

By Lemma 15, if $\rho(k_0, a) = 1$ or $\rho(k_0, b) = 1$, then $A(k_0+pj) \equiv 0 \pmod{p}$ for all j and thus $C_x(k_0 + pk_1) \equiv 0 \pmod{p}$. Otherwise $A(k_0 + pj) = A(k_0)B$, where $B \equiv A'(j) \pmod{p}$, and we have

$$\begin{aligned}
&C_x(k_0 + pk_1) \\
&\quad\equiv \sum_{j=0}^{k_1} A(k_0)A'(j)A'(k_1 - j)[D_{\theta(x)}(k_1 - j) - D_{\theta(x)}(j)] \\
&\quad\equiv A(k_0)\left[\sum_{j=0}^{k_1} A'(j)A'(k_1 - j)D_{\theta(x)}(k_1 - j) - \sum_{j=0}^{k_1} A'(j)A'(k_1 - j)D_{\theta(x)}(j)\right] \\
&\quad\equiv A(k_0)\left[\sum_{j'=0}^{k_1} A'(k_1 - j')A'(j')D_{\theta(x)}(j') - \sum_{j=0}^{k_1} A'(j)A'(k_1 - j)D_{\theta(x)}(j)\right] \\
&\quad\equiv 0 \pmod{p},
\end{aligned}$$

where we have used the substitution $j' = k_1 = j$ in the first sum. This holds for all x, and thus the theorem follows from (3.22). □

Lemma 17 (Dwork's Lemma)**.** *Let* $f(z) \in 1 + z\mathbb{Q}_p[[z]]$*. Then* $f(z) \in 1 + z\mathbb{Z}_p[[z]]$ *if and only if*
$f(z^p)/(f(z))^p \in 1 + pz\mathbb{Z}_p[[z]]$.

Proof. See [Kob84], Chapter IV, Lemma 3. □

We now prove a key corollary to Theorem 16.

Corollary 18. *If $a, b \in C_p$ and $\{\theta(a), \theta(b)\} = \{a, b\}$ then*

$$\exp\left(\frac{F_1(a,b;z)}{F(a,b,1;z)}\right) \in \mathbb{Z}_p[[z]]. \tag{3.28}$$

Proof. Dwork remarks that this is an "immediate consequence" of Theorem 16. Lian and Yau ([LY98], Corollary 6.7) and Zudilin ([Zud02], Lemma 5) provide proofs using Dwork's Lemma. For the special case we are concerned with here, let $f(z) = F_1(a,b;z)/F(a,b,1;z)$. Then $f(z) \in z\mathbb{Q}[[z]]$, and $\exp(f(z)) \in 1 + z\mathbb{Q}[[z]] \subseteq 1 + z\mathbb{Q}_p[[z]]$. By Theorem 16, $f(z^p) - pf(z) \in pz\mathbb{Z}_p[[z]]$, so let $f(z^p) - pf(z) = pg(z)$ where $g(z) \in z\mathbb{Z}_p[[z]]$. Then $\frac{\exp(f(z^p))}{\exp(f(z))^p} = \exp(f(z^p) - pf(z)) = \exp(pg(z)) \in 1 + pz\mathbb{Z}_p[[z]]$, since $\exp(pg(z)) = 1 + \sum_{k=1}^{\infty} \frac{p^k}{k!} g(z)^k$ and $v_p(k!) < \frac{k}{p-1} \leq k$ by (3.6). Now apply Dwork's Lemma to obtain (3.28). □

3.6 Main Theorem

We now prove the following special case of Conjecture 11.

Theorem 19. *Let $m \geq 3$. Let $J_m(z) = \sum_{n=-1}^{\infty} a_n q^n$. Then $a_n a_{-1}^n \in \mathbb{Z}_p$ for all odd primes $p \equiv 1 \pmod{4m}$.*

Proof. By Theorem 9 it suffices to prove that

$$\exp\left(\frac{F_1(\alpha,\beta;z)}{F(\alpha,\beta,1;z)}\right) \in \mathbb{Z}_p[[z]]. \tag{3.29}$$

By Corollary 18 it suffices to show that $\{\theta(\alpha), \theta(\beta)\} = \{\alpha, \beta\}$. From (3.3) we have

$$\alpha = \frac{m-2}{4m}, \quad \beta = \frac{m+2}{4m}. \tag{3.30}$$

Now $\{\theta(\alpha), \theta(\beta)\} = \{\alpha, \beta\}$ follows from Lemma 13. $\square$

It thus follows from Dirichlet's theorem on primes in arithmetic progressions that $a_n a_{-1}^n \in \mathbb{Z}_p$ for infinitely many primes p. Still this is much weaker than we should be able to prove. Let us make another conjecture.

Conjecture 20. *For p prime, if* $\exp\left(\frac{F_1(2\alpha,2\beta;z)}{F(2\alpha,2\beta,1;z)}\right) \in \mathbb{Z}_p[[z]]$ *then* $\exp\left(\frac{F_1(\alpha,\beta;z)}{F(\alpha,\beta,1;z)}\right) \in \mathbb{Z}_p[[z]]$.

Theorem 21. *Assume Conjecture 20 is true. Let $m \geq 3$.*
Let $J_m(z) = \sum_{n=-1}^{\infty} a_n q^n$. Then $a_n a_{-1}^n \in \mathbb{Z}_p$ for all odd primes $p \equiv \pm 1 \pmod{2m}$.

Proof. By Theorem 9 and Conjecture 20 it suffices to prove that

$$\exp\left(\frac{F_1(2\alpha, 2\beta; z)}{F(2\alpha, 2\beta, 1; z)}\right) \in \mathbb{Z}_p[[z]]. \tag{3.31}$$

By Corollary 18 it suffices to show that $\{\theta(2\alpha), \theta(2\beta)\} = \{2\alpha, 2\beta\}$. From (3.3) we have

$$2\alpha = \frac{m-2}{2m}, \quad 2\beta = \frac{m+2}{2m}. \tag{3.32}$$

If $p \equiv 1 \pmod{2m}$, then $\theta(2\alpha) = 2\alpha$ and $\theta(2\beta) = 2\beta$ by Lemma 13. If $p \equiv -1 \pmod{2m}$, then $\theta(2\alpha) = 2\beta$ and $\theta(2\beta) = 2\alpha$, since $-(m-2) \equiv m+2 \pmod{2m}$. In either case $\{\theta(2\alpha), \theta(2\beta)\} = \{2\alpha, 2\beta\}$ and the theorem is proven. $\square$

Note this is still not as strong as the conjecture, which claims $a_n a_{-1}^n \in \mathbb{Z}_p$ for all odd primes $p \equiv \pm 1 \pmod{m}$. Only when m is odd are the two are equivalent. It is interesting that it seems easier to prove the case $p \equiv \pm 1 \pmod{2m}$, which corresponds to the case in which p splits in $\mathbb{Z}[\lambda_m]$.

CHAPTER 4

Modular Forms for Hecke Groups

In this chapter we discuss the fourier coefficients of modular forms and cusp forms. In the first section we show how to define these forms in terms of the modular function J_m and what that implies for the properties of their coefficients. We then attempt to derive the Eisenstein series for the Hecke groups G_m directly, which would provide an alternate method to determine their coefficients. For the arithmetic cases $m = 3, 4, 6$ the coefficients can be described exactly, but even for the simplest non-arithmetic case $m = 5$ we run into difficulties related to characterizing the elements of G_5, an unresolved problem that has been studied since at least the 1950s. In the non-arithmetic cases the methods of the previous chapter seem the best way to determine information about the fourier coefficients.

4.1 Modular Forms and Cusp Forms

Following Hecke ([Hec38]), we say that f is a modular form of weight k for G_m if f is holomorphic on $\mathcal{H}^*$, $f(T(z)) = f(z)$ and $f(S(z)) = (-iz)^k \gamma f(z)$, where $\gamma = (-1)^{n_i}$ and n_i is the order of the zero at i. The weight k must be a rational number and its two smallest possible values are $4/(m-2)$ and $2m/(m-2)$. The number of linearly independent modular forms of weight k is less than or equal to $\left\lfloor \frac{k}{2}(\frac{1}{2} - \frac{1}{m}) \right\rfloor + 1 = \left\lfloor \frac{k(m-2)}{4m} \right\rfloor + 1$. Therefore there is at most one function (up to a constant factor) for each of these two smallest weights. Hecke proves that these functions exist and are given by

$$\tilde{f}_0(z) = \left(\frac{J_m'(z)^2}{J_m(z)(J_m(z)-1)}\right)^{1/(m-2)}$$

$$\tilde{f}_i(z) = \left(\frac{J_m'(z)^m}{J_m(z)^{m-1}(J_m(z)-1)}\right)^{1/(m-2)}.$$

where $\gamma = 1$ for $\tilde{f}_0$ and $\gamma = -1$ for $\tilde{f}_i$. Furthermore every modular form for G_m, for any weight, is a polynomial in $\tilde{f}_0$ and $\tilde{f}_i$.

Considering the Fourier expansion of J at $i\infty$, let $\hat{J}(q) = J(z)$. Then

$$\hat{f}_0(q) = \left(\left(\frac{2\pi i}{\lambda_m}\right)^2 \frac{q^2 \hat{J}_m'(q)^2}{\hat{J}_m(q)(\hat{J}_m(q)-1)}\right)^{1/(m-2)}$$

$$\hat{f}_i(q) = \left(\left(\frac{2\pi i}{\lambda_m}\right)^m \frac{q^m \hat{J}_m'(q)^m}{\hat{J}_m(q)^{m-1}(\hat{J}_m(q)-1)}\right)^{1/(m-2)}.$$

Multiplying by appropriate constants gives

$$f_0(q) = \left(\frac{q^2 \hat{J}_m'(q)^2}{\hat{J}_m(q)(\hat{J}_m(q)-1)}\right)^{1/(m-2)}$$

$$f_i(q) = \left((-1)^m \frac{q^m \hat{J}_m'(q)^m}{\hat{J}_m(q)^{m-1}(\hat{J}_m(q)-1)}\right)^{1/(m-2)}.$$

Both $f_0(q)$ and $f_i(q)$ have q-series starting with the constant term 1, and it follows that for $m = 3$ we have $f_0(q) = E_4(z)$ and $f_i(q) = E_6(z)$ near $i\infty$ (see the next section for the definition of the Eisenstein series E_k).

Fixing m let $K = K(q) = q\hat{J}_m(qA_m) = 1 + \sum_{n=1}^{\infty} c_n q^n$ where $c_n \in \mathbb{Q}$ for all n. Then

$$f_0(qA_m) = \left(\frac{(K - qK')^2}{K(K-q)}\right)^{1/(m-2)}$$

$$f_i(qA_m) = \left(\frac{(K - qK')^m}{K^{m-1}(K-q)}\right)^{1/(m-2)}.$$

We are interested in the (rational) coefficients of $F_j = \left(\frac{(K-qK')^j}{K^{j-1}(K-q)}\right)^{1/(m-2)}$ for $j = 2$ and $j = m$.

If the coefficients of K are p-adic integers for any prime p not dividing $m-2$, then the coefficients of F_j are also p-adic integers. The reason is as follows. Note that the numerator and denominator of F_j both have constant term 1. It follows from Lemma 5 that $\frac{(K-qK')^j}{K^{j-1}(K-q)} = 1 + qP(q)$ for some power series $P(q)$ and that all coefficients of $P(q)$ are p-adic integers. We then have

$$(1 + qP(q))^{1/(m-2)} = \sum_{n=0}^{\infty} \left(\frac{1}{m-2} - n + 1\right)_n \frac{(qP(q))^n}{n!}$$

and by Problem 138 of Part VIII, Chapter 3 of [PS98] the only additional primes that can divide the denominators of the coefficients of $(1 + qP(q))^{1/(m-2)}$ are divisors of $m-2$.

Note that the primes for which coefficients of K are p-adic integers in Conjectures and Theorems 11, 19 and 21 all satisfy the condition $p \nmid m-2$, and so p-adic results for K apply to the Eisenstein series as well.

We construct the canonical cusp form for G_m as the analog of the discriminant modular form:

$$\Delta(z) = E_4(z)^3 - E_6(z)^2.$$

It is a cusp form of weight 12. Viewing the forms as functions of q we can consider $\Delta(qA_m) = E_4(qA_m)^3 - E_6(qA_m)^2$, which will have rational integer coefficients. Trivially if the coefficients of both E_4 and E_6 are p-adic integers for any p, then the coefficients of Δ are p-adic integers for the same p, so we have information for cusp forms as well.

4.2 Eisenstein Series

The normalized Eisenstein series of even weight k for $G_3 = \Gamma(1) = PSL_2(\mathbb{Z})$ is

$$E_k(z) = \frac{1}{2\zeta(k)} \sum_{c,d\in\mathbb{Z}}{}' (cz+d)^{-k} = \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1}} (cz+d)^{-k} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \quad (4.1)$$

(see, for example, [Kob93], p. 111) where $\sum'$ means to exclude the case in which both $c = 0$ and $d = 0$ and where $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$ is the Riemann zeta function. Here $\sigma_k(n) = \sum_{d|n,d>0} d^k$ and B_k are the Bernoulli numbers defined by setting

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

For even $k > 0$ there is the identity ([Kob93], p. 110)

$$\zeta(k) = -\frac{(2\pi i)^k}{2(k!)} B_k. \quad (4.2)$$

The series for E_k converges absolutely for $k \geq 3$ and E_k is a modular form of weight k for G_3 for even $k \geq 4$, meaning

$$E_k(\gamma z) = (cz+d)^k E_k(z)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_3$. In particular $E_k(T(z)) = E_k(z)$ and $E_k(S(z)) = z^k E_k(z)$.

There is another way to interpret the Eisenstein series which leads to a generalization for the other Hecke groups (see [Leh64], p. 282 and [Iwa97], Chapter 3). Let $\Gamma_\infty = \langle T_3 \rangle$ be the subgroup of G_3 stabilizing ∞. Then the right cosets $\Gamma_\infty \gamma$, where $\gamma \in G_3$, consist of matrices all with the same bottom row. Furthermore if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\gamma' = \left(\begin{smallmatrix} A & B \\ c & d \end{smallmatrix}\right)$ have the same bottom row then they must be in the same coset, since $\gamma' \gamma^{-1}$ is upper-triangular and thus in Γ_∞. Also note that as γ and $-\gamma$ are equivalent in G_3, it follows a matrix with bottom row $(c \ d)$ and bottom row $(-c \ -d)$ are both in the same coset.

It follows

$$E_k(z) = \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1}} (cz+d)^{-k} = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} j_\gamma(z)^{-k}$$

where $j_\gamma(z) = cz + d$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. This function satisfies $j_{\alpha\beta}(z) = j_\alpha(\beta z) j_\beta(z)$. Now for general $\Gamma = G_m$ we have $\Gamma_\infty = \langle T_m \rangle$ (see [Shi94], Proposition 1.17). Define

$$E_k(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j_\gamma(z)^{-k}. \tag{4.3}$$

Then for $\alpha \in \Gamma$

$$\begin{aligned} E_k(\alpha z) &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j_\gamma(\alpha z)^{-k} \\ &= j_\alpha(z)^k \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j_{\gamma\alpha}(z)^{-k} \\ &= j_\alpha(z)^k E_k(z) \end{aligned}$$

showing E_k is a modular form of weight k. Note that as γ runs through a complete set of representatives of $\Gamma_\infty \setminus \Gamma$, so does $\gamma\alpha$, since if $\Gamma_\infty\gamma'$ is a coset then $\Gamma_\infty\gamma' = \Gamma_\infty\gamma\alpha$ where γ is such that $\gamma'\alpha^{-1} \in \Gamma_\infty\gamma$; also $\gamma'\gamma^{-1} \in \Gamma_\infty$ iff $\gamma'\alpha(\gamma\alpha)^{-1} \in \Gamma_\infty$.

The same argument as for G_3 shows that the cosets of $\Gamma_\infty \setminus \Gamma$ consist of exactly the matrices of Γ having the same bottom row (modulo multiplicative factor ± 1). We can further group these cosets into double cosets $\Gamma_\infty \setminus \Gamma / \Gamma_\infty$ in which each double coset (except Γ_∞ itself) is

$$\{\left(\begin{smallmatrix} * & * \\ c & d+nc\lambda \end{smallmatrix}\right) \in \Gamma : n \in \mathbb{Z}\} \tag{4.4}$$

where $\lambda = \lambda_m$, c and d are fixed, and $c > 0$ (cf. [Iwa97], Section 2.5). The Eisenstein series (4.3) then becomes

$$E_k(z) = 1 + \sum_{1\neq\gamma\in\Gamma_\infty\backslash\Gamma/\Gamma_\infty} \sum_{n\in\mathbb{Z}} (c(z+n\lambda)+d)^{-k} \tag{4.5}$$

Let $F(w) = \sum_{n\in\mathbb{Z}} f(w+n\lambda)$ where $f(w) = (c(z+w)+d)^{-k}$. Then $F(w)$ has period λ, and so has a Fourier expansion

$$F(w) = \lambda^{-1}\sum_{n\in\mathbb{Z}}\left(\int_0^\lambda \sum_{k\in\mathbb{Z}} f(x+k\lambda)e(-nx/\lambda)\,dx\right)e(nw/\lambda)$$

where $e(x) = e^{2\pi i x}$. Reversing the order of summation and integration, and making the change of variable $y = x + k\lambda$ as in the standard proof of Poisson summation results in

$$F(w) = \lambda^{-1}\sum_{n\in\mathbb{Z}}\left(\int_{-\infty}^{\infty} f(y)e(-ny/\lambda)\,dy\right)e(nw/\lambda).$$

Now let $w = 0$ to give

$$\sum_{n\in\mathbb{Z}}(c(z+n\lambda)+d)^{-k} = \lambda^{-1}\sum_{n\in\mathbb{Z}}\left(\int_{-\infty}^{\infty}(c(z+y)+d)^{-k}e(-ny/\lambda)\,dy\right).$$

Substituting into (4.5) gives

$$E_k(z) = 1 + \lambda^{-1}\sum_{1\neq\gamma\in\Gamma_\infty\backslash\Gamma/\Gamma_\infty}\sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty}(c(z+y)+d)^{-k}e(-ny/\lambda)\,dy.$$

Making the change of variable $v = y + z + d/c$, we get

$$E_k(z) = 1 + \lambda^{-1}\sum_{1\neq\gamma\in\Gamma_\infty\backslash\Gamma/\Gamma_\infty}\sum_{n\in\mathbb{Z}} e(nz/\lambda)e(nd/(c\lambda))\int_{-\infty+it}^{\infty+it}(cv)^{-k}e(-nv/\lambda)\,dv$$

where $z = s + it$, $t > 0$. Now following the argument in [Iwa97], pp. 50–51, we determine that

$$E_k(z) = 1 + \sum_{n=1}^{\infty} a_n q^n \tag{4.6}$$

where

$$a_n = \left(\frac{2\pi}{\lambda i}\right)^k \frac{n^{k-1}}{\Gamma(k)}\sum_{c>0} c^{-k}\sum_{\gamma=\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma/\Gamma_\infty} e\left(\frac{nd}{c\lambda}\right). \tag{4.7}$$

4.3 Eisenstein Series of Weight 4

Let us now concentrate on the case $k = 4$. For any G_m there is exactly one linearly independent modular form of weight 4 and it must be a constant multiple of

$$f_0(q)^{m-2} = \frac{q^2 \hat{J}'_m(q)^2}{\hat{J}_m(q)(\hat{J}_m(q) - 1)}. \tag{4.8}$$

Since this series starts with the constant term 1 as does the Eisenstein series of weight 4 given by (4.6), it follows the two must be equal.

We next derive the Eisenstein series using (4.7) in the arithmetic cases $m = 3, 4, 6$ and then attempt to derive the series in the simplest non-arithmetic case $m = 5$, pointing out the obstacles that arise.

4.3.1 The Case $m = 3$

The group G_3 is just $PSL_2(\mathbb{Z})$ and $\lambda_3 = 1$, and so by (4.4) each double coset with $c > 0$ corresponds to the pairs c, d with $(c, d) = 1$ and $0 \leq d < c$. It follows that the coefficients (4.7) of the Eisenstein series of even integer weight k are

$$\begin{aligned}
a_n &= \left(\frac{2\pi}{i}\right)^k \frac{n^{k-1}}{(k-1)!} \sum_{c>0} c^{-k} \sum_{\substack{0 \le d < c \\ (c,d)=1}} e\left(\frac{nd}{c}\right) \\
&= \left(\frac{2\pi}{i}\right)^k \frac{n^{k-1}}{(k-1)!} \sum_{c>0} c^{-k} \sum_{d|(c,n)} d\mu(c/d) \\
&= \left(\frac{2\pi}{i}\right)^k \frac{n^{k-1}}{(k-1)!} \sum_{d|n} d \sum_{\substack{c>0 \\ d|c}} c^{-k}\mu(c/d) \\
&= \left(\frac{2\pi}{i}\right)^k \frac{n^{k-1}}{(k-1)!} \sum_{d|n} d \sum_{C>0} (Cd)^{-k}\mu(C) \\
&= \left(\frac{2\pi}{i}\right)^k \frac{n^{k-1}}{\zeta(k)(k-1)!} \sum_{d|n} d^{1-k} \\
&= -\frac{2k}{B_k} \sum_{d|n} \left(\frac{n}{d}\right)^{k-1} \\
&= -\frac{2k}{B_k} \sigma_{k-1}(n)
\end{aligned}$$

where we have used the Ramanujan sum

$$\sum_{\substack{0 \le d < c \\ (c,d)=1}} e\left(\frac{nd}{c}\right) = \sum_{d|(c,n)} \mu(c/d)d,$$

the identity $\sum_{n=1}^{\infty} \mu(n)n^{-s} = 1/\zeta(s)$ (see [Mur01], Exercises 1.1.12 and 1.2.2), and (4.2). Note that this agrees with (4.1) and gives a precise description of the coefficients of the Eisenstein series for all even weights k.

4.3.2 The Cases $m = 4$ and $m = 6$

Both G_4 and G_6 are arithmetic groups. Here we have $\lambda_4 = \sqrt{2}$ and $\lambda_6 = \sqrt{3}$, and $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{3}]$ are both Euclidean domains. The matrices in G_m for $m = 4, 6$ are easy to describe.

Theorem 22. *The matrices in G_m for $m = 4, 6$ are those matrices in $SL_2(\mathbb{R})$ of the form*

$$\begin{pmatrix} a\lambda & b \\ c & d\lambda \end{pmatrix} \text{ and } \begin{pmatrix} a & b\lambda \\ c\lambda & d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{Z}$.

Proof. Detailed proofs are in [Hut02, You04]. They follow along the lines of similar algebraic proofs for the modular group and some congruence subgroups (see for example [DS05], Exercises 1.1.1 and 1.2.4). □

Since the determinants are $\lambda^2 ad - bc$ and $ad - \lambda^2 bc$ respectively, we see that given a pair of integers c, d with $(c, d) = 1$, there exists a matrix of the first type in G_m iff $\lambda^2 \nmid c$ and of the second type iff $\lambda^2 \nmid d$.

It follows that for $m = 4$ or $m = 6$ the coefficients (4.7) of the Eisenstein series of weight k are

$$a_n = \left(\frac{2\pi}{\lambda i}\right)^k \frac{n^{k-1}}{(k-1)!} \left(\sum_{\substack{c \in \mathbb{Z}^+ \\ (c,M)=1}} c^{-k} \sum_{\substack{0 \le d < c \\ (c,d)=1}} e\left(\frac{nd}{c}\right) + \sum_{c \in \mathbb{Z}^+} (c\lambda)^{-k} \sum_{\substack{0 \le d < Mc \\ (Mc,d)=1}} e\left(\frac{nd}{Mc}\right) \right)$$

where $M = \lambda^2$. For arbitrary $c, n, M \in \mathbb{Z}^+$, the Ramanujan sum

$$\sum_{\substack{0 \le d < Mc \\ (Mc,d)=1}} e\left(\frac{nd}{Mc}\right) = \sum_{d|(Mc,n)} \mu(Mc/d)d.$$

It follows that for M prime

$$\begin{aligned}
&\sum_{\substack{c\in\mathbb{Z}^+\\(c,M)=1}} c^{-k} \sum_{\substack{0\le d<c\\(c,d)=1}} e\left(\frac{nd}{c}\right) + \sum_{c\in\mathbb{Z}^+} (c\lambda)^{-k} \sum_{\substack{0\le d<Mc\\(Mc,d)=1}} e\left(\frac{nd}{Mc}\right)\\
&\quad= \sum_{\substack{c\in\mathbb{Z}^+\\(c,M)=1}} c^{-k} \sum_{d|(c,n)} \mu(c/d)d + \lambda^{-k} \sum_{c\in\mathbb{Z}^+} c^{-k} \sum_{d|(Mc,n)} \mu(Mc/d)d\\
&\quad= \sum_{d|n} d \sum_{\substack{c>0,(c,M)=1\\d|c}} c^{-k}\mu(c/d) + \lambda^{-k} \sum_{d|n} d \sum_{\substack{c>0\\d|Mc}} c^{-k}\mu(Mc/d)\\
&\quad= \sum_{d|n} d^{1-k} \sum_{\substack{C>0\\(dC,M)=1}} C^{-k}\mu(C) + \lambda^{-k} \sum_{d|n} d^{1-k}M^k \sum_{\substack{C>0\\M|dC}} C^{-k}\mu(C)\\
&\quad= \sum_{d|n} d^{1-k} \sum_{C>0} C^{-k}\mu(C) + (\lambda^k-1) \sum_{d|n} d^{1-k} \sum_{\substack{C>0\\M|dC}} C^{-k}\mu(C)\\
&\quad= \frac{\sigma_{k-1}(n)}{n^{k-1}\zeta(k)} + (\lambda^k-1) \sum_{d|n} d^{1-k} \sum_{\substack{C>0\\M|dC}} C^{-k}\mu(C)\\
&\quad= \frac{\sigma_{k-1}(n)}{n^{k-1}\zeta(k)} + (\lambda^k-1) \left(\sum_{\substack{d|n\\M|d}} d^{1-k}\frac{1}{\zeta(k)} + \sum_{\substack{d|n\\M\nmid d}} d^{1-k} \sum_{\substack{C>0\\M|C}} C^{-k}\mu(C) \right)\\
&\quad= \frac{\sigma_{k-1}(n)}{n^{k-1}\zeta(k)} + b_{n,M}(\lambda^k-1)\frac{\sigma_{k-1}(n/M)}{n^{k-1}\zeta(k)} + (\lambda^k-1)\frac{\sigma_{k-1}(N)}{N^{k-1}(1-M^k)\zeta(k)}\\
&\quad= \frac{\sigma_{k-1}(n)}{n^{k-1}\zeta(k)} + b_{n,M}(\lambda^k-1)\frac{\sigma_{k-1}(n/M)}{n^{k-1}\zeta(k)} - \frac{\sigma_{k-1}(N)}{(\lambda^k+1)N^{k-1}\zeta(k)}
\end{aligned}$$

where $N = n/M^{v_M(N)}$, $b_{n,M} = 1$ if $M|n$ and 0 otherwise, and where we have made the substitutions $C = c/d$ and $C = Mc/d$ respectively in the sums in the fourth line. To calculate $X = \sum_{\substack{C>0\\M|C}} C^{-k}\mu(C)$, let $Y = \sum_{\substack{C>0\\M\nmid C}} C^{-k}\mu(C)$. Then $\zeta(k)^{-1} = X + Y$, and $X = -M^{-k}Y$ (if M is prime). Solving for X we get $X = ((1-M^k)\zeta(k))^{-1}$.

Now let $k = 4$. We have

$$a_n = \frac{2^4\cdot 3\cdot 5}{M^2}\left(\sigma_3(n) + b_{n,M}(M^2-1)\sigma_3(n/M) - \frac{M^{3v_M(n)}\sigma_3(N)}{(M^2+1)}\right).$$

Since $\sigma_3(n) = \sigma_3(N)\sigma_3(M^{v_M(n)}) \geq M^{3v_M(n)}\sigma_3(N) > \frac{M^{3v_M(n)}\sigma_3(N)}{(M^2+1)}$, it follows $a_n > 0$. (The same argument holds for general k.)

For $M = 2$ it is clear $a_n \in \mathbb{Z}^+$. We can calculate the Eisenstein series for $m = 4$ to be

$$\begin{aligned} E_4(z) = 1 &+ 48q + 624q^2 + 1344q^3 + 5232q^4 + 6048q^5 + 17472q^6 + 16512q^7 \\ &+ 42096q^8 + 36336q^9 + 78624q^{10} + O(q^{11}) \end{aligned}$$

which agrees with a calculation using (4.8) (here $A_4 = 2^{-8}$).

For $M = 3$,

$$a_n = \frac{8}{3}\left(10\sigma_3(n) + 80b_{n,3}\sigma_3(n/3) - 3^{3v_3(n)}\sigma_3(N)\right).$$

If $3 \nmid n$ then $a_n = \frac{8}{3}(9\sigma_3(n))$, and if $3|n$ then
$10\sigma_3(n) + 80\sigma_3(n/3) = 10\sigma_3(N)(\sigma_3(3^{v_3(n)}) + 8\sigma_3(3^{v_3(n/3)})) \equiv 0 \pmod 3$. In either case $a_n \in \mathbb{Z}^+$. We can calculate the Eisenstein series for $m = 6$ to be

$$\begin{aligned} E_4(z) = 1 &+ 24q + 216q^2 + 888q^3 + 1752q^4 + 3024q^5 + 7992q^6 + 8256q^7 \\ &+ 14040q^8 + 24216q^9 + 27216q^{10}O(q^{11}). \end{aligned}$$

This also agrees with a calculation using (4.8) (here $A_6 = 2^{-2}3^{-3}$).

4.3.3 The Case $m = 5$

We now look even more specifically at the Eisenstein series of weight 4 for G_5. First note that $\lambda = \lambda_5 = 2\cos(\pi/5) = \frac{1}{2}(1+\sqrt{5})$. It is a root of $t^2 - t - 1$, the other root of which is $\lambda' = \frac{1}{2}(1-\sqrt{5})$; we have $\lambda + \lambda' = 1$, $\lambda\lambda' = -1$ and $\lambda^2 = \lambda + 1$.

Lemma 23. *$\mathbb{Z}[\lambda]$ is a Euclidean domain.*

Proof. We follow the standard proof technique as in [ER99], Chapter 2. The norm function is the usual one: $|N(a+b\lambda)| = |(a+b\lambda)(a+b\lambda')| = |a^2+ab-b^2|$. Given $\alpha, \beta \in \mathbb{Z}[\lambda]$, it follows $\alpha/\beta = r + s\lambda \in \mathbb{Q}[\lambda]$ and so choose $\gamma = m + n\lambda \in \mathbb{Z}[\lambda]$ such that $|r-m| \le 1/2$ and $|s-n| \le 1/2$. It then suffices to show that $|N(\alpha/\beta - \gamma)| < 1$, which follows since $|(r-m)^2 + (r-m)(s-n) - (s-n)^2| \le 1/2 < 1$. $\square$

By Dirichlet's Unit Theorem, the group of units of $\mathbb{Z}[\lambda]$ is the product of ± 1 and the infinite cyclic group generated by a single fundamental unit, chosen to be the minimal unit $a + b\lambda > 1$. If $a + b\lambda$ is a unit, then $a^2 + ab - b^2 = \pm 1$, and so $a = \frac{-b\pm\sqrt{5b^2\pm 4}}{2}$. The minimal positive value of $(a-1) + b\lambda = \frac{2b\lambda - b - 2 \pm\sqrt{5b^2\pm 4}}{2}$ is achieved when $b = 1$ and $a = 0$. It follows λ is the fundamental unit, and all units are of the form $\pm\lambda^k$ for $k \in \mathbb{Z}$. In terms of the $\mathbb{Z}$-basis $(1, \lambda)$ for $\mathbb{Z}[\lambda]$, we have $\lambda^k = F_k + F_{k+1}\lambda$ and $\lambda^{-k} = (-1)^k(F_{k+1} - F_k\lambda)$ for $k > 0$ where F_k is the kth Fibonacci number (defined so that $F_0 = 0$, $F_1 = 1$, and $F_k = F_{k-2} + F_{k-1}$ for $k \ge 2$).

The primes $\mathbb{Z}$ split in $\mathbb{Z}[\lambda]$ as follows (see [Mar77], p. 74). The only ramified prime is $5 = \sqrt{5}^2$, where $\sqrt{5} = 2\lambda - 1$. The prime 2 and all odd primes not congruent to $\pm 1 \bmod 5$ remain prime. Odd primes p congruent to $\pm 1 \bmod 5$ split into $p = (p, n + \sqrt{5})(p, n - \sqrt{5})$ where $n^2 \equiv 5 \pmod{p}$.

The residue A_5 of the Fourier series of J_5 can be calculated to be ([Ral63])

$$A_5 = \frac{\sqrt{5}(2+\sqrt{5})^{\sqrt{5}}}{2^6 5^3}. \tag{4.9}$$

The elements of G_5 have been investigated by Leutbecher ([Leu67], [Leu74]) and Rosen ([Ros54], [Ros86]). In particular Leutbecher ([Leu67]) proves that every element of $\mathbb{Q}(\lambda)$ is a cusp of some matrix of G_5. Furthermore it is easy to show that there is only one matrix in G_5 for each cusp. Since $\mathbb{Z}[\lambda]$ is a UFD, we can write the cusp as a/c where $a, c \in \mathbb{Z}[\lambda]$, $(a, c) = 1$, and this representation is unique up to units. If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $C = \left(\begin{smallmatrix} \mu a & B \\ \mu c & D \end{smallmatrix}\right)$ are two matrices in G_5 with the same cusp a/c (here μ is a unit), then $C^{-1}A \in \Gamma_\infty = \langle T_5 \rangle$. Since $C^{-1}A = \left(\begin{smallmatrix} Da-Bc & Db-Bd \\ 0 & \mu \end{smallmatrix}\right)$ it follows $\mu = \pm 1$.

The following is a proof of Leutbecher's theorem which fills in some details.

Theorem 24. *(Leutbecher) Let $\alpha \in \mathbb{Q}(\lambda)$. Then there exists some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_5$ such that $b/d = \alpha$.*

Proof. We present an algorithm for determining $M \in G_5$ such that $M(\alpha) = 0$. It follows $M^{-1}(0) = \alpha$ and if $M^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then $b/d = \alpha$. The algorithm is as follows. Given $\alpha \in \mathbb{Q}(\lambda)$, first apply T^n where n is the unique integer such that $-\lambda/2 < \alpha + n\lambda \leq \lambda/2$. If $\alpha + n\lambda = 0$ we are done; otherwise apply S to get $\alpha' = \frac{-1}{\alpha + n\lambda}$. Now repeat the process with α'. We must prove this algorithm always terminates.

Given $\alpha \in (-\lambda/2, \lambda/2]$, let $\alpha = r + s\lambda = \beta/\gamma$, where $r, s \in \mathbb{Q}$ and $\beta, \gamma \in \mathbb{Z}[\lambda]$, $(\beta, \gamma) = 1$. Note that r, s are unique and β, γ are unique up to units. Also r and γ are unchanged if we apply T^n to α. Let $p = |rN(\gamma)|$ and $q = |N(\gamma)|$. Then p, q are unchanged if T^n is applied to α or γ is multiplied by a unit. Furthermore p, q are both non-negative integers, since we can write $\alpha = \frac{\beta}{\gamma} \cdot \frac{\bar{\gamma}}{\bar{\gamma}} = \frac{m}{N(\gamma)} + \frac{n\lambda}{N(\gamma)} = r + s\lambda$ where $m, n \in \mathbb{Z}$ and $\bar{\gamma}$ is the conjugate of γ under $\lambda \mapsto \lambda'$. It follows $p = \pm m$. Letting $\alpha = \pm \frac{p + \lambda p'}{q}$, it follows $p' = \pm sq = \pm n$, also an integer.

Let $G(p, q) = 5p^2 + 4pq + 2q^2$. This is a positive definite quadratic form whose minimal positive value is 2, which occurs when $p = 0$ and $q = 1$. This is exactly the case in which $\alpha = 0$. In the case $\alpha \neq 0$ we want to prove that if p_1, q_1 are the nonnegative integers corresponding to $-1/\alpha$, then $G(p_1, q_1) < G(p, q)$. Given $\alpha = \pm \frac{p + \lambda p'}{q}$, we have $-1/\alpha = \pm \frac{q}{p + \lambda p'} = -\frac{\gamma}{\beta}$. Taking the absolute value of the norm of both sides gives $\frac{q^2}{|p^2+pp'-p'^2|} = \frac{q}{|N(\beta)|}$, from which we get $q_1 = |N(\beta)| = \frac{|p^2+pp'-p'^2|}{q}$. Writing $-1/\alpha = \pm\frac{q}{p+\lambda p'} = \pm\frac{q(p+p'-p'\lambda)}{|p^2+pp'-p'^2|}$ as $\pm\frac{p_1+\lambda p_1'}{q_1}$ we see that $p_1 = |p+p'|$.

We now consider two cases:

$$\text{I: } 0 \leq p + \lambda p' \leq \lambda q/2 \quad \text{and} \quad \text{II: } -\lambda q/2 < p + \lambda p' < 0$$

In case I we have

$$-p/\lambda \leq p' \leq q/2 - p/\lambda \quad \text{and}$$
$$p/\lambda^2 \leq p + p' \leq p/\lambda^2 + q/2,$$

from which follow

$$p_1 \leq p/\lambda^2 + q/2;$$
$$q_1 \leq (2\lambda - 1)p/2 - q/4 \qquad \text{if } q/2 \leq p/\lambda,$$
$$q_1 \leq p/2 + q/4 \qquad \text{if } p/\lambda < q/2.$$

If $q/2 \leq p/\lambda$ then $0 \leq p^2/\lambda^2 - pq/\lambda + q^2/4 \leq p'^2 \leq p^2/\lambda^2$. We also have $p^2/\lambda^2 \leq p^2 + pp' \leq p^2/\lambda^2 + pq/2$, and so combining the two we get $0 \leq p^2 + pp' - p'^2 \leq (1+2/\lambda)pq/2 - q^2/4$, which implies $q_1 \leq (1+2/\lambda)p/2 - q/4$ and since $1 + 2/\lambda = 2\lambda - 1$ we have the result.

If $p/\lambda < q/2$ then $-q/2 \leq p' \leq q/2 - p/\lambda$ implies $|p'| \leq q/2$, and so $p^2/\lambda^2 - q^2/4 \leq p^2 + pp' - p'^2 \leq pq/2 + q^2/4$ implies $q_1 \leq p/2 + q/4$.

In case II we have

$$-p/\lambda - q/2 < p' < -p/\lambda \quad \text{and}$$
$$p/\lambda^2 - q/2 < p + p' < p/\lambda^2,$$

from which follow

$$\begin{aligned} q_1 &\le (2\lambda - 1)p/2 + q/4; && \\ p_1 &\le p/\lambda^2 && \text{if } p + p' \ge 0, \\ p_1 &\le -p/\lambda^2 + q/2 && \text{if } p + p' < 0. \end{aligned}$$

For the q_1 case we have $p^2/\lambda^2 - pq/2 < p^2 + pp' < p^2/\lambda^2$ and $-p^2/\lambda^2 - pq/\lambda - q^2/4 < -p'^2 < -p^2/\lambda^2$ which combine to give $q_1 \le (1+2/\lambda)p/2 + q/4 = (2\lambda - 1)p/2 + q/4$.

In all cases we have inequalities of the form $p_1 \le ap + bq$ and $q_1 \le cp + dq$, for $a, b, c, d \in \mathbb{R}$. It follows $G(p_1, q_1) \le G(ap + bq, cp + dq)$. Letting $G(ap + bq, cp + dq) = up^2 + vpq + wq^2$, in all four cases a straightforward calculation shows that $u < 5$, $v < 4$ and $w < 2$. Since $p, q \ge 0$ it follows $G(p_1, q_1) < G(p, q)$. $\square$

This immediately implies every element of $\mathbb{Q}(\lambda)$ is a cusp of some matrix of G_5 since if $\left(\begin{smallmatrix} b & a \\ d & c \end{smallmatrix}\right) \in G_5$ then $\left(\begin{smallmatrix} b & a \\ d & c \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & -b \\ c & -d \end{smallmatrix}\right) \in G_5$. We are actually interested in the ratio d/c rather than a/c, but given $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ it follows $A^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ and so there is a 1-1 correspondence between the cusps and the ratios d/c; it follows that for every element of $\alpha \in \mathbb{Q}(\lambda)$ there exists some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_5$ such that $d/c = \alpha$, and if there are two such matrices then their bottom rows are the same (modulo ± 1).

Given $\alpha \in \mathbb{Q}(\lambda)$, we are interested in its unique representation d/c, where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_5$. Considering $A^{-1}S = \left(\begin{smallmatrix} -b & -d \\ a & c \end{smallmatrix}\right)$ it suffices to consider the unique representative of α in the orbit of 0. Let M^{-1} be the matrix in the proof of Leutbecher's theorem such that $M^{-1}(0) = \alpha$. Setting $M^{-1} = A^{-1}S$ gives us the numbers c and d we are seeking. Following the algorithm it is easy to see that if d/c is the unique representation of α, then $-d/c$ is the unique representation of $-\alpha$.

We now turn to the problem of determining the coefficients (4.7) of the Eisenstein series of weight 4 for G_5. First note that given a double coset $\{\left(\begin{smallmatrix} * & * \\ c & d+nc\lambda \end{smallmatrix}\right) \in \Gamma : n \in \mathbb{Z}\} \in \Gamma_\infty \setminus \Gamma/\Gamma_\infty$, the ratio of the bottom two entries of any element of a coset is $d/c + n\lambda$ for some $n \in Z$. It therefore suffices to take as representative for the coset the unique ratio in the range $(-\lambda/2, \lambda/2]$. Separating out the special cases 0 and $\lambda/2$ and rearranging the series for (4.7) gives

$$a_n = \left(\frac{2\pi}{\lambda i}\right)^k \frac{n^{k-1}}{\Gamma(k)} \left(1 + (-1)^n (2\lambda^2)^{-k} + \sum_{0<\alpha<\lambda/2} h(\alpha)^{-k} 2\cos(2\pi n\alpha/\lambda)\right) \quad (4.10)$$

where h is the "height" function such that $h(\alpha) = c$ where d/c is the unique representative of α. The problem thus becomes how to calculate $h(\alpha)$ given α.

Unfortunately it is not clear how to solve this problem. Nor does it seems possible to parameterize the elements of G_5 so that (4.7) can be used directly. Some efforts in this direction are described in the undergraduate research report [She07], and my own attempts at such parameterizations were also unsuccessful. Although not clear, it's possible there is no clean solution, and a reason might be that we did not factor the transcendental part of the coefficients out as we were able to do in the previous chapter. Indeed it is hard to see how one can naturally define the Eisenstein series and calculate it so that the transcendental part is factored out, without using the method of Section 4.1. Furthermore the fact that the coefficients are transcendental may well be intimately tied with the difficulty or impossibility of parameterizing the members of G_5. This indicates that the methods of the previous chapter can be used to give information about the fourier coefficients of modular forms and cusp forms in the case that other techniques may fail.

CHAPTER 5

Conclusion and Future Work

In this thesis we have observed experimentally the properties of primes dividing the denominators of fourier coefficients of triangle functions, formulated a conjecture describing these properties as precisely as possible, and proven part of the conjecture. The first area for future work is to prove the rest of the conjecture. However there is much more that can be done beyond this. The remainder of this chapter is devoted to describing other possible future directions and the connection of this work to other recent research on triangle functions.

If p is a prime for which the coefficients of J_m are p-adic integers, then the coefficients may satisfy further congruence properties, analogous to those satisfied by j ([Leh49a, Leh49b]). Akiyama has done some work in this area ([Aki92, Aki93]) but there is doubtless more to be discovered.

We have only examined triangle functions corresponding to Hecke groups; that is, we have limited ourselves to hyperbolic triangles in which two of the angles are fixed at 0 and $\pi/2$. An obvious generalization is to look at mappings from more general hyperbolic triangles to the upper half plane. The corresponding triangle groups have been classified by Takeuchi ([Tak77]).

There appears to be an interesting analogy between hauptmoduls for noncongruence subgroups of $SL_2(\mathbb{Z})$, first studied by Atkin and Swinnerton-Dyer ([AS71]) and hauptmoduls (in other words the J_m) for the non-arithmetic Hecke Groups.

In particular the coefficients of both appear to have non-bounded denominators, and there seems to be no genuine Hecke operators in either case. It would be worthwhile to explore this relationship further, and in particular to determine if any of the techniques used to explore non-congruence subgroups can be employed to study triangle functions, and vice-versa.

Finally there is some very interesting recent work on triangle groups being done by Wolfart and his collaborators. In particular Cohen and Wolfart ([CW90]) prove that there exist modular embeddings of triangle groups into arithmetic groups acting on products $\mathbb{H}^r$ of the upper half plane. The quotient spaces in this case are Hilbert modular spaces and the corresponding functions Hilbert modular forms. Schmidt ([Sch97]) gives an explicit embedding for the group G_5. Furthermore Schaller and Wolfart ([SW00]) define the notion of semi-arithmetic Fuchsian group, which encompasses all triangle groups. The idea is that these groups, although non-arithmetic in general, still have interesting arithmetic properties. The work of this thesis supports this idea. It appears from this work that Hilbert modular forms may well be the most natural and promising tool to both prove further properties of fourier coefficients of triangle functions, and to also explain them.

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