Math 32BH
Midterm 1 Solutions
February 2, 2005

1. (25 points)

   (a) (10 points) Give a precise geometric description of each of the cylindrical coordinates of a point $P \in \mathbb{R}^3$. Express each cylindrical coordinate in terms of the rectangular coordinates, noting any conditions that might apply. Express each rectangular coordinate in terms of the cylindrical coordinates, noting any conditions that might apply. Describe the restrictions you would place on the cylindrical coordinates so that every point in $\mathbb{R}^3$ has at least one representation in cylindrical coordinates, and as many points as possible have exactly one representation in cylindrical coordinates. Describe precisely the set of points that has more than one representation under your restrictions, and describe the set of all representations for each such point.

   The cylindrical coordinates of P are (r, θ, z), where r ($r \geq 0$) is the distance from P' (the projection of P onto the xy-plane) to the origin, θ is the angle ($\theta \in [0, 2\pi)$) that the line segment from the origin to P' makes with the positive x-axis, and z is the distance of the point to the xy plane. We have $r = \sqrt{x^2 + y^2}$, $z = z$, and $\theta = \tan^{-1} \frac{y}{x}$ for $x > 0$ and $y \geq 0$; $\theta = \tan^{-1} \frac{y}{x} + 2\pi$ for $x > 0$ and $y < 0$; and $\theta = \tan^{-1} \frac{y}{x} + \pi$ for $x < 0$. If $x = 0$ then $\theta = \pi/2$ if $y > 0$ and $\theta = 3\pi/2$ if $y < 0$; it is undefined if $y = 0$, see below. Conversely $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. The restrictions on r and θ already mentioned ensure every point not on the z axis has exactly one cylindrical coordinate tuple. Those points can be represented as $(0, \theta, z)$ for any $\theta \in [0, 2\pi)$.

   (b) (15 points) Give a precise geometric description of each of the spherical coordinates of a point $P \in \mathbb{R}^3$. Express each spherical coordinate in terms of the rectangular coordinates, noting any conditions that might apply. Express each rectangular coordinate in terms of the spherical coordinates, noting any conditions that might apply. Describe the restrictions you would place on the spherical coordinates so that every point in $\mathbb{R}^3$ has at least one representation in spherical coordinates, and as many points as possible have exactly one representation in spherical coordinates. Describe precisely the set of points that has more than one representation under your restrictions, and describe the set of all representations for each such point.

   The spherical coordinates of P are (ρ, ϕ, θ), where ρ ($\rho \geq 0$) is the distance from P to the origin, ϕ ($\phi \in [0, \pi]$) is the angle the line segment from the origin to P makes with the positive z axis, and θ is the angle ($\theta \in [0, 2\pi)$) that the line segment of the projection of P onto the xy plane makes with the x-axis. We have $\rho = \sqrt{x^2 + y^2 + z^2}$, $\phi = \cos^{-1} \frac{z}{\sqrt{x^2+y^2+z^2}}$ (for $(x, y, z) \neq (0, 0, 0)$; otherwise undefined as noted below) and $\theta = \tan^{-1} \frac{y}{x}$ for $x > 0$ and $y \geq 0$; $\theta = \tan^{-1} \frac{y}{x} + 2\pi$ for $x > 0$ and $y < 0$; and $\theta = \tan^{-1} \frac{y}{x} + \pi$ for $x < 0$. If $x = 0$ then $\theta = \pi/2$ if $y > 0$ and $\theta = 3\pi/2$ if $y < 0$; it is undefined if $y = 0$, see below. Conversely $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho\cos\phi$. The restrictions on ρ, ϕ and θ already mentioned ensure every point not on the z axis has exactly one spherical coordinate tuple. All points on the positive z axis can be represented as $(z, 0, \theta)$ for any $\theta \in [0, 2\pi)$. All points on the negative z axis can be represented as $(-z, \pi, \theta)$ for any $\theta \in [0, 2\pi)$. The origin can be represented as $(0, \phi, \theta)$ for any $\phi \in [0, \pi]$ and any $\theta \in [0, 2\pi)$.

2. (25 points) Find the volume of the region that lies inside both the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 - 2x = 0$.

   $4\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^{2\cos\theta} r\sqrt{4-r^2}\,dr\,d\theta = \frac{-32}{3}\int_0^{\pi/2}(\left|\sin^3\theta\right| - 1)d\theta = \frac{-32}{3}\int_0^{\pi/2}(\sin^3\theta - 1)d\theta = \frac{16}{3}\left(\pi - \frac{4}{3}\right)$. Note that if you make θ go from $-\pi/2$ to $\pi/2$ you have $\left|\sin^3\theta\right| \neq \sin^3\theta$ and need to deal with that.

3. (25 points) Evaluate $\iiint_E y\,dV$ where E is the interior of the pyramid with vertices $(0, 0, 0)$, $(0, 2, 0)$, $(2, 0, 0)$, $(2, 2, 0)$ and $(1, 1, 1)$.

   $\int_0^1\int_z^{2-z}\int_z^{2-z} y\,dx\,dy\,dz = \int_0^1\int_z^{2-z} 2y(1-z)\,dy\,dz = \int_0^1 (1-z)(4-4z)\,dz = \frac{4}{3}$.

4. (25 points)

(a) (10 points) Evaluate $\iiint_E xy^2z^3\,dV$ where E is the region bounded by the surface $z = xy$ and the planes $y = x$, $x = 1$, and $z = 0$.

$\int_0^1 \int_0^x \int_0^{xy} xy^2z^3\,dz\,dy\,dx = \frac{1}{364}$.

(b) (15 points) Find the volume of the region described by $x^2 + y^2 + z^2 \geq a$, $x^2 + y^2 + z^2 \leq b$, $x^2 + y^2 \leq \frac{z^2}{3}$, where $0 < a < b$.

Note that there are symmetric regions above and below the xy plane. The volume is thus

$2\int_0^{2\pi} \int_0^{\pi/6} \int_{\sqrt{a}}^{\sqrt{b}} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = 2\int_0^{2\pi} \int_0^{\pi/6} \int_{\sqrt{a}}^{\sqrt{b}} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta =$

$\frac{2}{3}(b^{3/2} - a^{3/2}) \int_0^{2\pi} \int_0^{\pi/6} \sin\phi\,d\phi\,d\theta = \frac{4\pi}{3}(b^{3/2} - a^{3/2})(1 - \frac{\sqrt{3}}{2})$.