Math 32BH
Homework 8 Solutions

I graded 4 of the problems:
Page 467: 14, 16;
Page 477: 4, 8.

The following are solutions to the homework problems and additional comments for the problems I graded. Note that solutions are often brief; if you need more detail please ask in section or office hours. I may well have made errors in my solutions so please let me know if I did. For grading information see my class webpage.

General Comments
The maximum number of points was 12. The high score was 12, the median was 11, and the mean was 10.1.

Page 467

2. Parameterize the top hemisphere as $(\sin u \cos v, \sin u \sin v, \cos u)$ where $(u, v) \in [0, \pi/2] \times [0, 2\pi]$. Then $X_u \times X_v = (\sin^2 u \cos v, \sin^2 u \sin v, \sin u \sin v)$, and it can be seen that the normal vector points outward (note that in Example 5.5 the normal vector points inward). Also $|X_u \times X_v| = \sin u$. The integral is then $\int_0^{2\pi} \int_0^{\pi/2} \sin^3 u \cos^2 u \, du \, dv = \frac{4\pi}{15}$.

4. Parameterize S as $(2\cos u, 2\sin u, v)$ where $(u, v) \in [-\pi, \pi] \times [0, 2\cos u + 2]$. Then the normal vector points outward and $|X_u \times X_v| = 2$. The integral is $\int_{-\pi}^{\pi} \int_0^{2\cos u+2} 2v^2 \, dv \, du = \frac{80\pi}{3}$.

10. The integral is $-\int_0^{\pi} \int_0^{2\pi} \sin^3 v \cos v (1 + \cos^2 u) \, du \, dv = 0$.

14. Parameterize as in exercise 4; then $X_u \times X_v = (2\cos u, 2\sin u, 0)$ and the integral is $\int_{-\pi}^{\pi} \int_0^{2\cos u+2} 4 \, dv \, du = 16\pi$.

16. Parameterize S as $(\cos u, \sin u, v)$ where $(u, v) \in [-\pi, \pi] \times [0, \cos u + 2]$. Then $X_u \times X_v = (\cos u, \sin u, 0)$ and the integral is $\int_{-\pi}^{\pi} \int_0^{\cos u+2} (2\cos^2 u - 3\sin^2 u) \, dv \, du = -2\pi$.

21. Answer is in the back of the book. Note that this is a special case of exercise 22.

22. There is a typo in the problem: $\left|\mathbf{x}^2\right|$ should be $|\mathbf{x}|^2$. Apparently from the solution to exercise 21 the flow is supposed to be from a source inside the sphere. So follow Example 6.5 except multiply by -1 (for the change in direction), and also $\frac{k}{a^2}$ (for the difference in the problem you are solving). The result is $4k\pi$, which is independent of the radius a.

Page 477

2. Since $\nabla \times F = 0$, by Stokes' Theorem the integral is 0.

4. By Stokes' Theorem integral around the unit square with vertices at the origin and $(1, 1)$ in the counterclockwise direction. Since $z = 0$ we have $F(x, y, z) = (y, 0, 0)$, and only the segment from $(0, 1)$ to $(1, 1)$ gives a nonzero integral. We thus get $-\int_0^1 dt = -1$. You can also use Green's Threom.

8. By Stokes' Theorem the integral is $\int_0^{2\pi} (\cos^2 t - \sin^3 t) \, dt = \pi$.

12. By Stokes' Theorem the integral is $\int_0^{2\pi} (\cos^2 t \sin t - \sin^3 t) \, dt = 0$.

16. First note $\nabla \times F = 0 = (2y - 2z, 2z - 2x, 2x - 2y)$. The curve of intersection can be parametrized in cylindrical coordinates as $(\cos u + 1, \sin u, \sqrt{2\cos u + 2})$ where $u \in [0, 2\pi]$. There are a number of ways to construct a surface with this curve as the boundary; one way is to parameterize using $X(u, v) = (v\cos u + 1, v\sin u, \sqrt{2v\cos u + 2})$ where $(u, v) \in [0, 2\pi] \times [0, 1]$. Then $X_u = \left(-v\sin u, v\cos u, -\frac{v\sin u}{\sqrt{2v\cos u+2}}\right)$ and $X_v = \left(\cos u, \sin u, \frac{v\cos u}{\sqrt{2v\cos u+2}}\right)$. Note that this parameterization is only smooth on the interior of the surface, and in fact the derivative blows up on the boundary; although the book is vague about this I believe this is okay, and don't see a way to parametrize the surface so that the partials don't blow up on the boundary.

    We get $X_u \times X_v = \left(\frac{v}{\sqrt{2v\cos u+2}}, 0, -v\right)$ and so our norm is pointing down (by the instructions it must point up); to compensate we'll just multiply the integral by -1. The result is then $-\int_0^1 \int_0^{2\pi} 2\left(v\sin u - \sqrt{2v\cos u + 2}, \sqrt{2v\cos u + 2} - v\cos u - 1, v\cos u - 1 - v\sin u\right) \cdot \left(\frac{v}{\sqrt{2v\cos u+2}}, 0, -v\right) du\, dv = -2\int_0^1 \int_0^{2\pi} \left(\frac{v^2 \sin u}{\sqrt{2v\cos u+2}} - v^2\cos u + v^2 \sin u\right) du\, dv$. This is easily evaluated to get 0 (!!!), which makes me think there must be an easier way to do this, but I don't see it. Note that I may well have made a mistake somewhere as well.

17. Just follow the hint.

21. Part (a) is an easy calculation, and part (b) follows from Stokes' Theorem.