Math 32BH
Homework 6 Solutions

I graded 4 of the problems:
Page 436: 4, 26;
Page 448: 8, 16.

The following are solutions to the homework problems and additional comments for the problems I graded. Note that solutions are often brief; if you need more detail please ask in section or office hours. I may well have made errors in my solutions so please let me know if I did. For grading information see my class webpage.

General Comments
The maximum number of points was 12. The high score was 12, the median was 12, and the mean was 11.5.

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2. $\operatorname{div} F(x) = 2x_1 + x_1x_3x_4 - x_4 + x_1$; $\operatorname{div} F(x_0) = 5$.

4. $\operatorname{div} F(x) = \frac{x^2+y^2+z^2+2(xy-xz+yz)}{(x^2+y^2+z^2)^2}$; $\operatorname{div} F(x_0) = \frac{9}{25}$.

8. $\operatorname{curl} F(x) = (0,0,0)$; $\operatorname{curl} F(x_0) = (0,0,0)$.

10. We have $\nabla \frac{f}{g} = \left(\frac{\partial f/g}{\partial x_1}, \ldots, \frac{\partial f/g}{\partial x_n} \right)$ and $\frac{g\nabla f - f \nabla g}{g^2} = \left(\frac{g\frac{\partial f}{\partial x_1} - f\frac{\partial g}{\partial x_1}}{g^2}, \ldots, \frac{g\frac{\partial f}{\partial x_n} - f\frac{\partial g}{\partial x_n}}{g^2} \right)$ and so the theorem follows by the one-variable theorem applied to each coordinate.

14. A tedious but straightfoward calculation.

17. See exercise 14 solution.

26. We have
$\int_{\partial D} f(\mathbf{x})(\nabla g \cdot \mathbf{N})\, ds = \int_{\partial D} (f(\mathbf{x})\nabla g) \cdot \mathbf{N}\, ds = \iint_D \operatorname{div}(f(\mathbf{x})\nabla g)\, dA = \iint_d f(\mathbf{x})\nabla^2 g(\mathbf{x}) + \nabla f \cdot \nabla g\, dA$ by Theorem 3.4(a) (linearity), Exercise 7.2.28, and Theorem 3.4(b), and the result follows.

36. See exercise 14 solution.

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2. If $\frac{\partial p}{\partial x} = y^2$, then $p(x,y) = xy^2 + f(y)$, where f is some function of y alone. So if $\frac{\partial p}{\partial y} = 2xy + f'(y) = 2xy$, then $f'(y) = 0$. It follows $p(x,y) = xy^2 + C$ for some constant C. So the integral evaluates to $p(2,4) - p(0,0) = 32$.

6. If $\frac{\partial p}{\partial x} = 3x^2 \sin y + \cos^2 y$, then $p(x,y) = x^3 \sin y + x\cos^2 y + f(y)$, where f is some function of y alone. So if $\frac{\partial p}{\partial y} = x^3 \cos y - 2x \cos y \sin y + f'(y) = -x(\sin 2y - x^2 \cos y)$, then $f'(y) = 0$. It follows $p(x,y) = x^3 \sin y + x \cos^2 y + C$ for some constant C. So the integral evaluates to 0.

8. If $\frac{\partial p}{\partial x} = x^2 + yz$, then $p(x,y,z) = \frac{1}{3}x^3 + xyz + f(y,z)$, where f is some function of y and z alone. So if $\frac{\partial p}{\partial y} = xz + \frac{\partial f(y,z)}{\partial y} = y^2 + xz$, then $f(y,z) = \frac{1}{3}y^3 + g(z)$, where g is a function of z alone. A similar argument shows $g(z) = \frac{1}{3}z^3 + C$, where C is a constant. It follows $p(x,y,z) = \frac{1}{3}(x^3 + y^3 + z^3) + xyz + C$. So the integral evaluates to $p(1,1,1) - p(0,0,0) = 2$.

12. If $\frac{\partial p}{\partial x} = 3y^2 + 6xy$, then $p(x,y) = 3xy^2 + 3x^2y + f(y)$, where f is some function of y alone. So if $\frac{\partial p}{\partial y} = 3x^2 + 6xy + f'(y) = 3x^2 + 6y$, there is no way to choose $f(y)$ to make this true, and so no such potential function exists. This can also be proven by computing the Jacobian and seeing that it is not symmetric.

16. If $\frac{\partial p}{\partial x} = 2xy^2$, then $p(x,y,z) = x^2y^2 + f(y,z)$, where f is some function of y and z alone. So if $\frac{\partial p}{\partial y} = 2yx^2 + \frac{\partial f(y,z)}{\partial y} = x^2z^3$, there is no way to choose $f(y,z)$ to make this true, and so no such potential function exists. This can also be proven by computing the Jacobian and seeing that it is not symmetric.

18. If $\frac{\partial p}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}$, then $p(x, y, z) = \sqrt{x^2 + y^2 + z^2} + f(y, z)$, where f is some function of y and z alone. By symmetry we can see that $p(x, y, z) = \sqrt{x^2 + y^2 + z^2} + C$ works for any constant C, and there are no singularities in D.

21. As $\nabla p = f$ where $p(\mathbf{x}) = -k\frac{1}{|\mathbf{x}|}$ (see Example 3.7), it follows from Theorem 4.1 that the work done is $-k\frac{1}{|\mathbf{x_1}|} + k\frac{1}{|\mathbf{x_0}|} \approx k\frac{1}{|\mathbf{x_0}|}$ since $|\mathbf{x_1}|$ is large.

22. Since $|x_0| = R$ it follows from exercise 21 that the work is approximately $-gR$.