Midterm 2 Solutions

1. **(15 points)** Determine the length of the curve

$$y = x^{3/2}$$

from the point $(0,0)$ to the point $(4,8)$.

Solution. We have $y = f(x) = x^{3/2}$, and $f'(x) = \frac{3}{2}\sqrt{x}$. Plugging into the formula for arc length gives

$$\int_0^4 \sqrt{1+(9/4)x}\,dx = \frac{8}{27}\left(1+\frac{9}{4}x\right)^{3/2}\Bigg]_0^4 = \frac{8}{27}\left(10^{3/2}-1\right).$$

2. **(15 points)** A ball is dropped from a height of 10 meters onto a hard, level surface, and each time it rebounds its height is $\frac{2}{3}$ the high point since the last bounce. Assuming the ball continues to bounce indefinitely, what is the total distance it travels?

Solution. The series representing the distance is

$$10 + 2\left(\frac{2}{3}\right)10 + 2\left(\frac{2}{3}\right)^2 10 + \cdots = 10 + 20\left(\frac{2}{3}\right)\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1} = 10 + \frac{40}{3}\frac{1}{1-\frac{2}{3}} = 50.$$

See also exercise 52 in section 12.2, assigned for homework.

3. **(20 points)**

   (a) **(10 points)** Prove that $n! \ge 2^{n-1}$ for all $n \ge 1$ by mathematical induction.

   Solution. Base case: For n=1 we have $1! = 1 \ge 1 = 2^0$.

   Induction: Assume true for n and prove for $n+1$. We have $(n+1)! = (n+1)n!$, and by the induction hypothesis $n! \ge 2^{n-1}$, and so $(n+1)! \ge (n+1)2^{n-1}$. Furthermore $n+1 \ge 2$ for all $n \ge 1$, and therefore $(n+1)! \ge 2^n$ which is what we wanted to prove.

   (b) **(10 points)** Determine whether or not

$$\sum_{n=1}^{\infty}\frac{1}{n!}$$

   converges. Justify your answer. If it converges, give upper and lower bounds on its value.

   Solution. By part (a),

$$\frac{1}{n!} \le \frac{1}{2^{n-1}}$$

   for all $n \ge 1$, and so by the comparison test we have

$$\sum_{n=1}^{\infty}\frac{1}{n!} \le \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} = \frac{1}{1-\frac{1}{2}} = 2.$$

   Therefore the series converges and an upper bound on its value is 2. Since the series has all positive terms an obvious lower bound on its value is the first term 1. These bounds are sufficient for full credit, although one can do better of course. This problem is very similar to exercise 29 of section 12.5.

4. **(20 points)**

(a) **(10 points)** Give the definition of a geometric series whose first term is 1. When exactly does this series converge and what value does it converge to? You do not need to prove your answer.

Solution. Given $r \in \mathbb{R}$, a geometric series whose first term is 1 is

$$1 + r + r^2 + r^3 + \cdots = \sum_{n=1}^{\infty} r^{n-1} = \sum_{n=0}^{\infty} r^n.$$

Any of these three characterizations is fine. The series converges if and only if $|r| < 1$ and if it converges the value is

$$\frac{1}{1-r}.$$

(b) **(10 points)** Give the formula for the the nth partial sum s_n of a geometric series whose first term is 1. Prove your formula is correct using either the textbook's method or mathematical induction. Be sure your proof is clear and careful.

Solution. *Textbook Method.* For $r = 1$, we have $s_n = n$ by definition. If $r \neq 1$, then

$$\begin{aligned} s_n &= 1 + r + \cdots + r^{n-1} \\ 1 + rs_n &= 1 + r + \cdots + r^{n-1} + r^n \\ &= s_n + r^n \end{aligned}$$

which, solving for s_n, implies

$$s_n = \frac{1-r^n}{1-r}.$$

Mathemetical Induction. For $r = 1$, we have $s_n = n$ by definition. For $r \neq 1$, we want to prove

$$s_n = \frac{1-r^n}{1-r}.$$

for all $n \geq 1$.

Base case. If $n = 1$ then $s_n = 1$ by definition, and $s_n = \frac{1-r}{1-r} = 1$ by the formula.

Induction. Assume true for n and prove for $n+1$. By the definition, $s_{n+1} = s_n + r^n$. By the induction hypothesis, $s_n = \frac{1-r^n}{1-r}$. Therefore

$$\begin{aligned} s_{n+1} &= \frac{1-r^n}{1-r} + r^n \\ &= \frac{1-r^n}{1-r} + \frac{r^n - r^{n+1}}{1-r} \\ &= \frac{1-r^{n+1}}{1-r} \end{aligned}$$

which is what we wanted to prove.

5. **(30 points)**

(a) **(5 points)** Define the p-series and carefully state for which real p the series converges and diverges. You do not need to justify your answer.

Solution. For $p \in \mathbb{R}$ the p-series is defined to be

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

It converges if and only if $p > 1$.

(b) **(5 points)** Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

What can you say about the convergence of this series by comparing it to p-series?

Solution. You can't say anything, since

$$\frac{1}{n \ln n} \leq \frac{1}{n}$$

for all $n \geq 2$ (the inequality would have to be reversed to show the series diverges using the comparison test), and

$$\frac{1}{n \ln n} \geq \frac{1}{n^p}$$

for all $p > 1$ and all sufficiently large n, since $\ln n$ grows more slowly than any positive power of n. Again the inequality would have to be reversed to show the series converges using the comparison test.

(c) **(10 points)** Determine whether or not

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

converges. Carefully justify your answer.

Solution. Since the comparison test fails, we try the integral test. The function $f(x) = \frac{1}{x \ln x}$ is clearly positive on $[2, \infty)$ and also decreasing since

$$f'(x) = \frac{-(1 + \ln x)}{(x \ln x)^2} < 0$$

on that interval. We have

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \int_{\ln 2}^{\infty} \frac{1}{u}\, du = \ln u]_{\ln 2}^{\infty} = \infty$$

using the substitution $u = \ln x$, and so the integral and series both diverge. This is exercise 21 of section 12.3.

(d) **(10 points)** Determine for what real q the series

$$\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^q}.$$

converges. Carefully justify your answer.

Solution. For $q \leq 1$ the series diverges by comparison to the series in part (c). So say that $q > 1$ and again use the integral test. The function $f(x) = \frac{1}{x(\ln x)^q}$ is clearly positive on $[2, \infty)$ and also decreasing since

$$f'(x) = \frac{-(q(\ln x)^{q-1} + (\ln x)^q)}{(x(\ln x)^q)^2} < 0$$

on that interval. We have

$$\int_2^{\infty} \frac{1}{x(\ln x)^q}\, dx = \int_{\ln 2}^{\infty} u^{-q}\, du = \frac{u^{1-q}}{1-q}\bigg]_{\ln 2}^{\infty} = \frac{1}{(\ln 2)^{q-1}(q-1)}$$

using the substitution $u = \ln x$, and so the integral and series both converge for $q > 1$. This is exercise 25 of section 12.3.