Math 31B
Homework 9
Due Friday, March 16, 2007

Textbook Exercises to hand in

- **12.10:** 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 28, 30, 31, 32, 40, 42, 48, 50, 52, 54, 56, 58, 60, 62.
- **12.11:** 2, 4.
- **12.12:** 4, 6, 8, 10.

Additional Exercises to hand in

1. **Coefficients of Taylor Series.** Prove carefully by mathematical induction that if $f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$ then

$$f^{(n)}(x) = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} c_k (x-a)^{k-n}.$$

   Conclude that $c_n = \frac{f^{(n)}(a)}{n!}$ for all $n \geq 0$.

2. **Alternating Harmonic Series Revisited.** We know that the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges and it is interesting to know just what value it converges to. In fact we proved in exercise 36 of Section 12.5 that it converges to $\ln 2$. Review that exercise, and then prove that its value is $\ln 2$ by this different method.

   (a) Write down the Maclaurin series you derived for $\ln(1+x)$ in exercise 6 of Section 12.10, and also its interval of convergence.

   (b) Let $f(x) = \ln(1+x)$. Prove by mathematical induction that $f^{(n+1)}(x)$, the $n+1$st derivative of $\ln(1+x)$, is $(-1)^n \frac{n!}{(1+x)^{n+1}}$ for all $n \geq 0$.

   (c) The note at the top of page 800 of the text briefly mentions the integral form of the remainder term:

$$R_n(x) = \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t)\, dt.$$

   Using this formula and the fact that $1/(1+t)^{n+1} \leq 1$ for all $t \geq 0$ and $n \geq 0$, prove that $|R_n(1)| \leq \frac{1}{1+n}$. Why can't we use Taylor's Inequality (as it is stated in the textbook) to prove the same result?

   (d) Conclude that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$.

Suggested warm-up exercises (do not hand these in)

- **12.10:** Selected odd exercises from 3 to 29; 39, 41; 47 to 59.
- **12.11:** 1, 3.
- **12.12:** 3, 5, 7, 9.