Math 31B
Homework 8
Due Wednesday, March 7, 2007

Textbook Exercises to hand in

- **12.8:** 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 30.
- **12.9:** 3, 4, 6, 8, 10, 12, 14, 16, 18, 24, 26, 32, 35, 36, 38.

Additional Exercises to hand in

1. **Midterm Revisited, Part 1.** For problem 3 of the midterm, several people argued that $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges by comparing it to the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. They didn't justify this, but it does work, so let's do it carefully now.

   First prove that $n! \geq n^2$ for all $n \geq 4$ (note that it's true for $n = 1$ but fails for $n = 2$ and $n = 3$) by using mathematical induction. Then invoke the comparison test to conclude $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges. Note that the upper bound on the sum you get in this case is not as good as if you use a geometric series (as in the midterm solutions). What upper bound do you get in this case?

2. **Midterm Revisited, Part 2.** Solve problem 5 of the midterm again, doing it completely and carefully. You can of course refer to the solutions but it's best to try it without looking at them and refer to them only if you get stuck. Then, solve problem 26 of Section 12.3.

3. **Sums and Integrals.** In class we derived the formula

   $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

   and proved it by induction. In this problem we look at another important sum, $\sum_{k=1}^{n} k^2$. It is much harder to derive a formula for this sum, but we can actually use calculus to help!

   (a) First notice that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$, and that $\int_0^n x\, dx = \frac{n^2}{2}$. As we might expect, it looks like the integral is a good approximation to the sum.
   Plot x^2 (from 0 to n) and $\sum_{k=1}^{n} k^2$ (as a Riemann sum) on the same graph. Which has a greater value, $\int_0^n x^2\, dx$ or $\sum_{k=1}^{n} k^2$? By comparing to the integral, guess that $\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + an^2 + bn + c$, where a, b, c are real numbers. Plug in $n = 0, 1, 2$ into both sides of this equation to get three equations in the three unknowns a, b, c, and solve for them (note that this is similar to solving for unknown constants when you do integration by partial fractions).

   (b) You now have a formula for $s(n) = \sum_{k=1}^{n} k^2$, but you still have to prove it is actually correct. Prove the formula is correct by mathematical induction on n.

   (c) Note that in the equation $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, the left hand side must be an integer, whereas it looks like the right hand side could be a fraction. But this is never the case because either n or $n+1$ is always even and thus divisible by 2. Similarly your formula for $\sum_{k=1}^{n} k^2$ must always give an integer but it looks like it might sometimes produce a fraction as well. Why does it always produce an integer?

Suggested warm-up exercises (do not hand these in)

- **12.8:** Selected odd exercises from 3 to 29.
- **12.9:** Selected odd exercises from 5 to 25.