Final Exam Solutions

1. **(5 points)** Find

$$\lim_{x\to 0} \frac{1-\cos x}{x^2}.$$

Solution. Use l'Hôpital's Rule. We have

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} \stackrel{H}{=} \lim_{x\to 0} \frac{\sin x}{2x} \stackrel{H}{=} \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

This is problem 27 of section 7.7.

2. **(5 points)** Determine the radius of convergence and interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt[4]{n}}.$$

Solution. This is exercise 11 of section 12.8. By the ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}x^{n+1}}{\sqrt[4]{n+1}} \cdot \frac{\sqrt[4]{n}}{2^n x^n}\right| \to 2\,|x|$$

as $n \to \infty$, and so the radius of convergence is $\frac{1}{2}$. When $x = -\frac{1}{2}$, the series diverges by comparison to p-series ($p = \frac{1}{4}$); when $x = \frac{1}{2}$, the series converges by the Alternating Series Test. Therefore the interval of convergence is $(-\frac{1}{2}, \frac{1}{2}]$.

3. **(10 points)**

   (a) **(5 points)** Evaluate

$$\int \sin^3 x \, dx.$$

Solution. We have

$$\int \sin^3 x \, dx = \int (1-\cos^2 x)\sin x \, dx = \int -(1-u^2)\, du = \frac{u^3}{3} - u + C = \frac{1}{3}\cos^3 x - \cos x + C$$

using the substitution $u = \cos x$.

   (b) **(5 points)** Evaluate

$$\int_{-1}^{\sqrt{3}} \sqrt{4-t^2}\, dt.$$

Solution. Let $t = 2\sin x$, where $-\pi/2 \le x \le \pi/2$ (note that $\cos x \ge 0$ for all such x). Then $dt = 2\cos x$, and the limits are $\sin^{-1}(-1/2) = -\pi/6$ and $\sin^{-1}(\sqrt{3}/2) = \pi/3$. We have

$$\int_{-1}^{\sqrt{3}} \sqrt{4-t^2}\, dt = \int_{-\pi/6}^{\pi/3} 4\cos^2 x \, dx = \int_{-\pi/6}^{\pi/3} (2 + 2\cos 2x)\, dx = \pi + [\sin(2\pi/3) - \sin(-\pi/3)] = \pi + \sqrt{3}.$$

4. **(10 points)** Evaluate

$$\int \frac{2x^2 + x + 4}{x^3 + 4x}\,dx.$$

Solution. Use integration by partial fractions. The degree of the numerator is less than the denomoninator, and the denominator factors as $x(x^2+4)$ where x^2+4 is irreducible. We have

$$\frac{2x^2 + x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}.$$

Multiply both sides by $x(x^2+4)$, resulting in

$$2x^2 + x + 4 = A(x^2+4) + (Bx + C)x = (A + B)x^2 + Cx + 4A$$

This immediately gives us $A = 1$, $B = 1$ and $C = 1$. We thus get

$$\begin{aligned}\int \frac{2x^2 + x + 4}{x^3 + 4x}\,dx &= \int \frac{1}{x}\,dx + \int \frac{x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx \\ &= \ln|x| + \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}(x/2) + K.\end{aligned}$$

This is almost identical to example 5 of section 8.4.

5. **(10 points)**

   (a) **(5 points)** Give the definition of a geometric series whose first term is 1. When exactly does this series converge and what value does it converge to? You do not need to prove your answer.

   Solution. Given $r \in \mathbb{R}$, a geometric series whose first term is 1 is

$$1 + r + r^2 + r^3 + \cdots = \sum_{n=1}^{\infty} r^{n-1} = \sum_{n=0}^{\infty} r^n.$$

   Any of these three characterizations is fine. The series converges if and only if $|r| < 1$ and if it converges the value is

$$\frac{1}{1-r}.$$

   (b) **(5 points)** Give the formula for the the nth partial sum s_n of a geometric series whose first term is 1. Prove your formula is correct using either the textbook's method or mathematical induction. Be sure your proof is clear and careful.

   Solution. *Textbook Method.* For $r = 1$, we have $s_n = n$ by definition. If $r \neq 1$, then

$$\begin{aligned} s_n &= 1 + r + \cdots + r^{n-1} \\ 1 + rs_n &= 1 + r + \cdots + r^{n-1} + r^n \\ &= s_n + r^n \end{aligned}$$

   which, solving for s_n, implies

$$s_n = \frac{1 - r^n}{1 - r}.$$

   Mathemetical Induction. For $r = 1$, we have $s_n = n$ by definition. For $r \neq 1$, we want to prove

$$s_n = \frac{1 - r^n}{1 - r}.$$

   for all $n \geq 1$.

   Base case. If $n = 1$ then $s_n = 1$ by definition, and $s_n = \frac{1-r}{1-r} = 1$ by the formula.

Induction. Assume true for n and prove for $n+1$. By the definition, $s_{n+1} = s_n + r^n$. By the induction hypothesis, $s_n = \frac{1-r^n}{1-r}$. Therefore

$$\begin{aligned} s_{n+1} &= \frac{1-r^n}{1-r} + r^n \\ &= \frac{1-r^n}{1-r} + \frac{r^n - r^{n+1}}{1-r} \\ &= \frac{1-r^{n+1}}{1-r} \end{aligned}$$

which is what we wanted to prove.

6. **(10 points)**

(a) **(5 points)** For the following subparts, you need only write the answer. You do not need to justify anything. Your answer must be perfectly correct, however, to get the point.

(i) **(1 point)** State the general form of the Maclaurin series for the function $f(x)$.
Solution. $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.

(ii) **(1 point)** State the general form of the Taylor series about point a for the function $f(x)$.
Solution. $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$.

(iii) **(1 point)** Write down the Maclaurin series for e^x.
Solution. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(iv) **(1 point)** Write down the Maclaurin series for $\cos x$.
Solution. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

(v) **(1 point)** Write down the Maclaurin series for $\sin x$.
Solution. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

(b) **(5 points)** Assuming $f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$, carefully derive, step-by-step, formulas for c_0, c_1, c_2 and c_3 in terms of f and its derivatives.

Solution. Given $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$, plugging in $x = 0$ we get $c_0 = f(0)$.
Taking a derivative we have $f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$, and $c_1 = f'(0)$.
Taking another derivative we have $f''(x) = 2c_2 x + 6c_3 x + 12c_4 x^2 + \cdots$, and $c_2 = f''(0)/2$.
Taking one more derivative we have $f^{(3)}(x) = 6c_3 + 24c_4 x + \cdots$, and $c_3 = f^{(3)}(0)/6$.

7. **(10 points)**

(a) **(5 points)** Determine for what real p the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

converges. Carefully justify your answer.

Solution. This is the p-series, convergent for $p > 1$ and divergent for $p \le 1$. We prove these facts as was done in the textbook (section 12.3). First if $p < 0$, then $\lim_{n\to\infty}(1/n^p) = \infty$, and if $p = 0$, then $\lim_{n\to\infty}(1/n^p) = 1$. In either case the series diverges by the Test for Divergence. If $p > 0$, then the function $f(x) = 1/x^p$ is continuous, positive, and decreasing on $[1, \infty)$. Therefore we can use the integral test. For $p = 1$, $\int_1^\infty 1/x = \ln x]_1^\infty = \infty$, and so the integral and series both diverge. For $p > 0$ and $p \ne 1$, $\int_1^\infty 1/x^p = \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\Big]_1^t = \lim_{t\to\infty}\left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\right)$, which converges to $\frac{1}{p-1}$ for $p > 1$ and diverges to ∞ if $p < 1$. The result follows.

(b) **(5 points)** Determine for what real p the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}.$$

converges. Carefully justify your answer.

Solution. This is exercise 25 of section 12.3 and problem 5(d) of the second midterm. For $p < 1$ the series diverges by comparison to the harmonic series. So say that $p \geq 1$ and use the integral test. The function $f(x) = \frac{1}{x(\ln x)^p}$ is clearly positive on $[2, \infty)$ and also decreasing since $x(\ln x)^p$ is increasing for $p \geq 1$. For $p = 1$, we have

$$\int_2^\infty \frac{1}{x \ln x}\, dx = \int_{\ln 2}^\infty \frac{1}{u}\, du = \ln u\Big]_{\ln 2}^\infty = \infty$$

using the substitution $u = \ln x$, and so the integral and series both diverge for $p = 1$.
For $p > 1$ we have

$$\int_2^\infty \frac{1}{x(\ln x)^p}\, dx = \int_{\ln 2}^\infty u^{-p}\, du = \left.\frac{u^{1-p}}{1-p}\right]_{\ln 2}^\infty = \frac{1}{(\ln 2)^{p-1}(p-1)}$$

using the same substitution, and so the integral and series both converge for $p > 1$.

8. **(10 points)** Find the Maclaurin series for $f(x) = -\ln(1-x)$ using the definition of Maclaurin series. Show all of your work. What is the radius of convergence and the interval of convergence for the series?

Solution. It is easy to show (by making a table and seeing the pattern, or by mathematical induction) that $f^{(n)}(x) = (n-1)!(1-x)^{-n}$ for all $n \geq 1$. Therefore $f(0) = 0$ and $f^{(n)}(0) = (n-1)!$ for all $n \geq 1$. It follows

$$f(x) = \sum_{n=1}^\infty \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots .$$

By the ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right| \to |x|$$

as $n \to \infty$, and so the radius of convergence is 1. When $x = 1$ we have the harmonic series (plus 1) which doesn't converge, and when $x = -1$ we have the alternating harmonic series (plus 1) which does converge, and so the interval of convergence is $[-1, 1)$.

9. **(10 points)**

(a) **(5 points)** Let $c_0 = C$, where C is some constant, and for $n > 0$ define

$$c_{2n} = \frac{c_{2n-2}}{2n}.$$

Prove that

$$c_{2n} = \frac{C}{2^n n!}$$

for all $n \geq 0$ using mathematical induction.

Solution. Base case, $n = 0$. By definition we have $c_0 = C = \frac{C}{2^0 0!}$.

Induction. Assume true for n, prove for $n+1$. We have $c_{2n+2} = \frac{c_{2n}}{2n+2} = \frac{C}{2^n n!} \cdot \frac{1}{2(n+1)} = \frac{C}{2^{n+1}(n+1)!}$ using the induction hypothesis.

(b) **(5 points)** Solve the differential equation $y' = xy$ by assuming $y = \sum_{n=0}^\infty c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ is a power series and solving for the coefficients c_n. Derive a general formula for c_n. Can you tell what function this power series represents?

Solution. Letting $y = \sum_{n=0}^\infty c_n x^n$ we have $y' = \sum_{n=0}^\infty n c_n x^{n-1}$. Then $y' = xy$ implies $\sum_{n=0}^\infty n c_n x^{n-1} = \sum_{n=0}^\infty c_n x^{n+1}$. Subtracting the right hand side from both sides gives $c_1 + \sum_{n=1}^\infty ((n+1)c_{n+1} - c_{n-1})x^n = 0$. It follows $c_n = 0$ for all n odd, and letting $c_0 = C$, some constant, we have $(2n)c_{2n} = c_{2n-2}$, and so we have the same recurrence relation as in part (a). Using that result it follows

$$y = C \sum_{n=0}^\infty \frac{x^{2n}}{2^n n!}.$$

You should recognize this as the power series for $Ce^{\frac{x^2}{2}}$, which if you try it out solves the differential equation.

10. **(20 points)** Although $n! = 1 \cdot 2 \cdots n$ is initially defined only for positive integers, Euler discovered a way to "extend" the factorial function to all positive real numbers. In this problem we will explore this function, called the Gamma (Γ) function.

For $x > 0$ a real number, define

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt.$$

This is an improper integral, but it is possible to prove it converges for all positive x. Assume that it converges for this problem.

(a) **(10 points)** Using integration by parts, prove

$$\Gamma(x+1) = x\Gamma(x)$$

for all real $x > 0$. Make sure to carefully handle any limits that arise.

Solution. We have

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x \, dt.$$

Let $u = t^x$ and $dv = e^{-t}\, dt$, so $du = xt^{x-1}\, dt$ and $v = -e^{-t}$. Then integration by parts gives

$$\Gamma(x+1) = -t^x e^{-t}\Big]_0^\infty + \int_0^\infty x e^{-t} t^{x-1} \, dt = x \int_0^\infty e^{-t} t^{x-1} \, dt = x\Gamma(x)$$

since $\lim_{t\to\infty} t^x e^{-t} = 0$. Why is this limit zero? Use l'Hôpital's Rule:

$$\lim_{t\to\infty} \frac{t^x}{e^t} \overset{H}{=} \lim_{t\to\infty} \frac{xt^{x-1}}{e^t}$$

and repeating enough times we eventually get either 0 or t to a negative power in the numerator, and in either case the limit is 0.

(b) **(10 points)** Evaluate $\Gamma(1)$. Then, using mathematical induction, prove that

$$\Gamma(n+1) = n!$$

for all integers $n \geq 0$.

Solution. From the definition, we have

$$\Gamma(1) = \int_0^\infty e^{-t} t^{1-1} \, dt = \int_0^\infty e^{-t} \, dt = -e^{-t}\Big]_0^\infty = 1.$$

Now prove $\Gamma(n+1) = n!$ by induction on n. The base case is $n = 0$, and we just showed $\Gamma(1) = 1 = 0!$. For induction, assume true for n and prove for $n+1$. In part (a) above we proved $\Gamma(x+1) = x\Gamma(x)$. Letting $x = n+1$, this means $\Gamma(n+2) = (n+1)\Gamma(n+1)$. By the induction hypothesis, $\Gamma(n+1) = n!$, and so $\Gamma(n+2) = (n+1)n! = (n+1)!$, which is what we wanted to show.