Math 115A
Homework 9 Comments

I graded 8 of the problems:
Section 6.2: 7, 12, 14
Section 6.3: 3a, 6, 11, 13a, 13b

Each problem is worth 2 points. A grade of 0 indicates no solution or a substantially wrong solution. A grade of 2 indicates a correct or nearly correct solution. Otherwise the grade given is 1.

Graded homework will be available in the box outside my office after Monday, December 8 at 10am. Please pick it (and any older homework) up before the final exam. If you have any questions about the problems or grading, please ask me in my office hours on Tuesday. After the final it will be too late! After grades are turned in I will throw out (recycle) any unclaimed homework.

The following are comments and occasionally solutions for the graded problems.

General Comments
The maximum number of points was 16. The high score was 15. The mean was 8.8.

6.2

7. I went over this problem in section, and those who just used my solution got full credit. Many people did this problem before section and used their own answer, which is great, except a lot of the answers were simply wrong! It's a good idea to go over problems you solved that were covered in section and make sure that your answer actually makes sense in light of what we talked about. Anyway here is the solution given in section, in case you mised it.

   ($\Longrightarrow$) If $z \in W^\perp$ then by definition $\langle z, v\rangle = 0$ for all $v \in W$, so since $\beta \subseteq W$ the result follows.

   ($\Longleftarrow$) Assume for concreteness that W is finite dimensional, so $\beta = \{v_1, \ldots, v_n\}$. Let $v \in W$. Then we can write $v = \sum_{i=1}^n a_i v_i$, where the $a_i \in \mathbb{F}$. Then $\langle z, v\rangle = \langle z, \sum_{i=1}^n a_i v_i\rangle = \sum_{i=1}^n \overline{a_i}\langle z, v_i\rangle$ by linearity. Since $\langle z, v_i\rangle = 0$ for all i (by hypothesis), it follows $\langle z, v\rangle = 0$.

12. This was again a problem I went over in section, and I wanted to see if people seemed to understand the proof. I'm not sure if I learned much though since almost everyone used my proof verbatim. Here it is again.

   To show $(R(L_{A^*}))^\perp = N(L_A)$, we prove set inclusion in both directions. First $N(L_A) \subseteq (R(L_{A^*}))^\perp$. Let $x \in N(L_A)$; then (using matrix notation throughout the problem), $Ax = 0$. We want to show $x \in (R(L_{A^*}))^\perp$, that is that x is orthogonal to every vector in the range of A^*. In symbols, we want to show that $\langle x, A^*y\rangle = 0$ for all $y \in \mathbb{F}^m$. However $\langle x, A^*y\rangle = \langle Ax, y\rangle = \langle 0, y\rangle = 0$ using the fact that $Ax = 0$.

   We now show $(R(L_{A^*}))^\perp \subseteq N(L_A)$. Again if $x \in (R(L_{A^*}))^\perp$ then $\langle x, A^*y\rangle = 0$ for all $y \in \mathbb{F}^m$. We want to show $Ax = 0$. Since $\langle x, A^*y\rangle = \langle Ax, y\rangle$, if $\langle Ax, y\rangle = 0$ for all $y \in \mathbb{F}^m$ then it must follow that $Ax = 0$ (see the solutions for problem 9 of section 6.1).

14. For the first equation we again prove set inclusion in both directions. If $x \in (W_1 + W_2)^\perp$, then $\langle x, w_1 + w_2\rangle = 0$ for all pairs $w_1 \in W_1$ and $w_2 \in W_2$. In particular $\langle x, w_1\rangle = 0$ for all $w_1 \in W_1$ (take $w_2 = 0$), and similarly $\langle x, w_2\rangle = 0$ for all $w_2 \in W_2$. It follows $x \in W_1^\perp \cap W_2^\perp$. Conversely if $x \in W_1^\perp \cap W_2^\perp$ then $\langle x, w_1\rangle = 0$ for all $w_1 \in W_1$ and $\langle x, w_2\rangle = 0$ for all $w_2 \in W_2$, so for all pairs $w_1 \in W_1$ and $w_2 \in W_2$ we have $\langle x, w_1 + w_2\rangle = \langle x, w_1\rangle + \langle x, w_2\rangle = 0$ by linearity.

   For the second equation we use the hint. Replacing W_1 and W_2 in the first equation with $W_1^\perp$ and $W_2^\perp$ respectively, we get $(W_1^\perp + W_2^\perp)^\perp = (W_1^\perp)^\perp \cap (W_2^\perp)^\perp = W_1 \cap W_2$. Since these are equal the orthogonal complements of both sides are equal, which is exactly the result we want.

   Quite a few people did the second part by starting with $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$ and taking the orthogonal complement of both sides, giving $(W_1 + W_2) = (W_1^\perp \cap W_2^\perp)^\perp$. They then claimed that the right hand side equals $(W_1^\perp)^\perp \cap (W_2^\perp)^\perp$ with no justification. In fact it is not true that $(A \cap B)^\perp = A^\perp \cap B^\perp$; a simple counterexample is to let A be the x-axis and B be the y-axis in $\mathbb{R}^2$.

6.3

3a. I was thinking of trying to grade part (c), but it seemed too painful, and looking through the homeworks almost no one was able to get it anyway. So I just graded part (a), which was too easy, but that's okay. The key points were to note that the standard basis β for $\mathbb{R}^2$ is orthonormal, so if we write $[T]_\beta = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$ we can use Theorem 6.10 to conclude that $[T^*]_\beta = [T]^*_\beta = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$ since the matrix happens to be self-adjoint (those were three key points if you were keeping count). It's then simple to calculate that $\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 11 \\ -12 \end{pmatrix}$, the answer in the back of the book.

6. Almost everyone got this, a straightforward application of parts (a), (c) and (d) of Theorem 6.11. For reference the solutions are $(T + T^*)^* = T^* + T^{**} = T^* + T = T + T^*$ and $(TT^*)^* = T^{**}T^* = TT^*$. A few people just said $(TT^*)^* = TT^*$ with no intermediate step. I feared they might be thinking they could apply (d) directly instead of using both (c) and (d), so I took off a point for this.

11. At first I thought this problem was wrong as stated since I'd forgotten T_0 represents a specific linear transformation (the zero transformation). I thought it just represented an arbitrary linear transformation, much as S_0 is used to represent an arbitrary set. Blame the textbook for inconsistent notation. Anyway no one else seemed to be confused, and almost everyone got at least the first half of this problem. If $T * T = T_0$, then $\langle T * T(x), x\rangle = 0$ for all $x \in V$. Then $\langle T * T(x), x\rangle = \langle T(x), T(x)\rangle = 0$ for all $x \in V$, and so it must be true that $T(x) = 0$ for all $x \in V$; in other words $T = T_0$.

    For the case TT^* people had more trouble. Some used the same argument as above to show $T^* = T_0$, but that wasn't what was asked for. There were however several good solutions. One way to proceed is to show that $T^* = T_0$, then invoke problem 13(b) of this section to show that $\text{rank}(T) = 0$, and recall from the true/false section of the second midterm that this means $T = T_0$. You can also argue that if $T^* = T_0$ then $\langle T^*(x), y\rangle = \langle x, T(y)\rangle = 0$ for all $x, y \in V$, and so it must follow that $T(y) = 0$ for all $y \in V$ and thus $T = T_0$.

13a. Almost everyone proved this as set inclusion in both directions, and did each direction the same way as the solution that follows. The inclusion $N(T) \subseteq N(T^*T)$ is the easy direction: If $x \in N(T)$ then $T(x) = 0$ and so certainly $T^*T(x) = 0$. Surprisingly a few people neglected to do this part, only proving the other direction, and they lost a point. To show the other inclusion, if $T^*T(x) = 0$ then consider $0 = \langle T^*T(x), x\rangle = \langle T(x), T(x)\rangle$. It must thus be the case that $T(x) = 0$. I didn't grade the second part of this problem ("Deduce that....") which is a straightforward application of the dimension theorem. Therefore those who just did the second part didn't get any credit for the problem.

13b. This problem was quite tough, and in fact only person got it right; no one else even managed partial credit. As with 13(a) I didn't grade the "Deduce from..." part, which was straightfoward.

    The difficulty here is that it's not obvious how to relate T and T^*. For example problem 7 of this section shows that $N(T) \neq N(T^*)$ in general, so that kind of approach isn't going to work. However if we look at problem 12(a) we see a ray of hope: It claims that $R(T^*)^\perp = N(T)$. Although this was not assigned it is essentially the same as problem 12 of section 6.2, which was assigned and which is solved above. So let's use it to prove this result. By the dimension theorem we have $\dim N(T) + \dim R(T) = \dim V$. So using the result of problem 12(a) gives

$$\dim R(T^*)^\perp + \dim R(T) = \dim V. \tag{1}$$

    By Theorem 6.7(c) we also know $\dim R(T^*) + \dim R(T^*)^\perp = \dim V$, so $\dim R(T^*)^\perp = \dim V - \dim R(T^*)$. Substituting into equation (1) gives the desired result $\dim R(T) = \dim R(T^*)$.