Math 115A
Homework 6 Comments

I graded 6 of the problems:
Section 2.5: 3b, 6a, 11
Section 5.1: 3b, 4g, 9

Each problem (except two of them; see below) is worth 2 points. A grade of 0 indicates no solution or a substantially wrong solution. A grade of 2 indicates a correct or nearly correct solution. Otherwise the grade given is 1.

If you believe a problem was misgraded, or I made some addition or other error, please write a short note explaining the situation, attach it to your homework, and return it to me (either in person, in my mailbox, or under my office door). I'll take a look and afterwards leave your homework in a box outside my office.

The following are comments and occasionally solutions for the graded problems.

General Comments
Since 3b and 4g from Section 5.1 were quite long I made them worth 4 points each instead of 2. I graded those plus 4 other problems, so the maximum number of points was 16. The high score was 16; yes for the first time this term someone finally got a perfect homework score! The mean was 9.2.
Since the homework was heavier on computation and lighter on proofs this time, I think many people found it a refreshing change. On the other hand many people didn't seem to realize how long the assignment was and apparently ran out of time at the end. Make sure you practice calculating change of basis, eigenvalues and eigenvectors so that you'll be able to do them quickly and confidently on tests.

2.5

3. I graded just part (b), whose answer was not in the back of the book. Part (b) was just a minor variation on part (a), and a good check of whether you understand what's going on. Here everything is the same as part (a) except the basis β is ordered differently. So now the first element of β' is (a_0, a_1, a_2) in coordinates of the ordered basis β. The other two are similar, and the answer is thus:
$$\begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}.$$
Most people got this right although a few didn't. I gave full credit for a correct answer, although I was happier if you indicated your reasoning. A wrong answer was generally worth 0 unless you showed some understanding and just made a mistake.

6. I graded just part (a). The answer was in the back of the book, and I gave full credit if you wrote down the answer and showed some work. Almost everyone got full credit, then, but it wasn't clear how many really understood what they were doing. The point here is that we're using Theorem 2.23. Let γ be the standard basis on F^2. Then A is a matrix representing a linear tranformation in γ coordinates; in symbols $A = [L_A]_\gamma$. To calculate $[L_A]_\beta$, we first have to change coordinates from β to γ, then calculate $[L_A]_\gamma$ (which expects an input in γ coordinates and whose output is also in γ coordinates) and then change basis back to β coordinates. So, using matrices, Q is the change of basis from β to γ, which is easy to calculate since γ is just the standard basis for F^2. To change from γ back to β we just calculate Q^{-1}.

11. In section I gave the hint to write down what Q and R are as matrices and then look back for relevant theorems that could help you out. So this was a good exercise to see if you understand what those theorems mean and how to use them. Most people got part (a), but most had trouble with part (b). First note that $Q = [I]_\alpha^\beta$ and $R = [I]_\beta^\gamma$, where $I : V \to V$ is the identity operator on V. Now by Theorem 2.11 it follows $RQ = [I]_\beta^\gamma [I]_\alpha^\beta = [II]_\alpha^\gamma = [I]_\alpha^\gamma$, since $II = I$ as a linear operator on V. Now note that $[I]_\alpha^\gamma$ is just the change of coordinate matrix from α-coordinates to γ-coordinates.

    For part (b), the relevant theorem is Theorem 2.18. First the theorem says that $Q = [I]_\alpha^\beta$ is invertible if I is invertible, but the identity operator is of course invertible with itself as its inverse. So that means Q^{-1} exists and by the theorem $Q^{-1} = ([I]_\alpha^\beta)^{-1} = [I]_\beta^\alpha$, which is the change of coordinate matrix from β-coordinates to α-coordinates.

5.1

3. I graded just part (b), whose answer was not in the back of the book. This was worth 4 points. Most people tried this, and the success rate was a lot higher than for 4g (below) for some reason. Quite a few people got it right. To find the eigenvalues we calculate det $A - \lambda I$, which results in a nasty cubic equation $t^3 - 6t^2 + 11t - 6 = 0$. I don't know an easy general method for finding the roots of cubics, but it looked like most people just guessed one solution λ and then divided the polynomial by $t - \lambda$ which results in a quadratic which is easy to factor. For example $t = 1$ is easily seen to be a solution, and dividing $t^3 - 6t^2 + 11t - 6$ by $t - 1$ we get $t^2 - 5t + 6$ which clearly factors as $(t - 2)(t - 3)$. So the three eigenvalues are $\{1, 2, 3\}$. I took off a point if you didn't show your work in how you derived the eigenvalues.

   Next we compute the null spaces corresponding to each eigenvalue, and the calculations are all unpleasant row reductions so I'll spare you the details, but each turns out to have dimension 1 and eigenvectors for $1, 2, 3$ respectively are $(1, 1, -1)^t$, $(1, -1, 0)^t$ and $(1, 0, -1)^t$ (any multiple of each of these is also fine).

   A couple people asked me if we have to prove these form a basis for $\mathbb{R}^3$, and the answer is that at this point you should. It turns out that if you have n distinct eigenvalues for $\mathbb{R}^n$ then their eigenvectors will always form a basis for $\mathbb{R}^n$, but this is not exactly obvious. In fact it's part of the next section in the textbook. The easiest way to show they are a basis is to prove they are linearly independent; then since there are 3 they must also span $\mathbb{R}^3$. I didn't take off any points for those who didn't show they were a basis, however.

   Finally Q is the matrix whose columns are these three eigenvectors, and D is the diagonal matrix whose diagonal entries are 1,2,3. If you calculate Q^{-1} and then $Q^{-1}AQ$ you should get D. It's not required for this problem, but is a good check to make sure you didn't make any mistakes. Of course you could make a mistake in the final inverse computation or matrix multiplication and then get worried for no reason, so maybe it's not worth it unless you're asked to do it.

4. I graded just part (g), whose answer was not in the back of the book. This was worth 4 points. A fair number of people attempted this, and a fair number of those indicated that they knew what they were doing, but there were a lot of opportunities to make mistakes in this problem and only two people got the correct answer (or very close). Two others came pretty close. When I tried this problem myself I made a number of mistakes, so I sympathize completely. But make sure to work out several of these problems and practice being careful for the exam.

   The first order of business is to pick an ordered basis for V and represent T as a matrix. The standard basis is of course $\{1, x, x^2, x^3\}$. You can also use the ordering $\{x^3, x^2, x, 1\}$ which will give a lower-triangular matrix instead of upper-triangular. Using $\gamma = \{1, x, x^2, x^3\}$, we evaluate T on each basis element in order to determine the columns of $[T]_\gamma$. You'll find the values to be $T(1) = -1$, $T(x) = x - 2$, $T(x^2) = 2x^2 - 2$ and $T(x^3) = 3x^3 + 6x - 8$. It follows

$$[T]_\gamma = \begin{pmatrix} -1 & -2 & -2 & -8 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

   Now there are two things to be careful about here. Two people put the columns in the wrong order. You have to use a consistent order of the basis for both the rows and columns! You won't even get the eigenvalues in this case. Another person wrote out the solutions of $T(x^i)$ as rows instead of columns. This still gives the right eigenvalues (the matrix is just transposed), but it completely confuses things when computing eigenvectors.

   Since this matrix is upper triangular (lower triangular would have been fine too) by problem 9 below we know the eigenvalues are simply the diagonal entries, which are $\{-1, 1, 2, 3\}$. So the next step is to substitute each into the matrix $[T]_\gamma - \lambda I$ and find a basis for the null space, which turns out to have dimension 1 in each case. Four possible basis vectors (multiples of these are fine too) are $(1, 0, 0, 0)^t$, $(1, -1, 0, 0)^t$, $(2, 0, -3, 0)^t$ and $(-1, 6, 0, 2)^t$. Now to complete the solution you should translate these back to vectors in V, since that was what was asked for, although I didn't take off for not doing this. The result is thus $\{1, 1 - x, 2 - 3x^2, -1 + 6x + 2x^3\}$.

9. Most people got this. The eigenvalues of M are the roots of $\det(M - tI)$, but since M is upper triangular so is $M - tI$. Now by property 4 of a determinant in Section 4.4, the determinant of $M - tI$ is the product of its diagonal entries. Say M is $n \times n$, and denote the entries of M as a_{ij}. Then the diagonal entries of $M - tI$ are $a_{ii} - t$ for $1 \leq i \leq n$. It follows $\det(M - tI) = (a_{11} - t)(a_{22} - t) \cdots (a_{nn} - t)$, and therefore the eigenvalues are simply $a_{11}, a_{22}, \ldots, a_{nn}$.