Math 115A
Homework 5 Comments

I graded 5 of the problems:
Section 2.4: 3cd, 11, 15, 20, 23

Each problem is worth 2 points. A grade of 0 indicates no solution or a substantially wrong solution. A grade of 2 indicates a correct or nearly correct solution. Otherwise the grade given is 1.

If you believe a problem was misgraded, or I made some addition or other error, please write a short note explaining the situation, attach it to your homework, and return it to me (either in person, in my mailbox, or under my office door). I'll take a look and afterwards leave your homework in a box outside my office.

The following are comments and occasionally solutions for the graded problems.

General Comments
Since I only graded 5 problems, the maximum number of points was 10. The high score was 9 and the mean 5. Since this was a short homework but had some fairly difficult problems on it, I concentrated more on the difficult problems this time. I was pleased to find that there were quite a few good solutions to these problems.
After the midterm, with this homework I see the class dividing more clearly into three groups (although there are a few people who are tough to classify): those who are on top of the material and are doing well; those who are putting in effort but are struggling; and those who seem to have just given up. I estimate about 1/3 of the class in the first group, 1/2 in the second group, and 1/6 in the third group.

2.4

3. I graded just parts (c) and (d); each was worth 1 point, and I gave a point if you said anything reasonable. Almost everyone got full credit, but surprisingly not everyone. This problem was a test to see if you understand the meaning of Theorem 2.19 and how to use it, so review it if you're unsure. For part (c) both dimensions are 4 so they are isomorphic. For part (d) the dimension of V is 3, as we calculated way back in problem 15 of section 1.6. Some people kindly included a basis for V. Therefore it cannot be isomorphic to $\mathbb{R}^4$.

11. Example 5 says to use the Lagrange interpolation formula to prove that T is 1-1; I'm not sure if this was covered in class or not. I guess not, though, since no one used it. There were three different good ways to solve this that I saw. The first was just to write out explicitly what T does to an arbitrary polynomial $f = ax^3 + bx^2 + cx + d \in P_3(\mathbb{R})$, which gives a system of 4 equations in 4 unknowns, and then solve this equation to show that $a = b = c = d = 0$.

    The second method was to have T act on the four basis elements of $P_3(\mathbb{R})$ and then show that these span $M_{2\times 2}(\mathbb{R})$; then invoke the dimension formula to conclude that $N(T) = \{0\}$. To show that $T(1), T(x), T(x^2), T(x^3)$ span $M_{2\times 2}(\mathbb{R})$ you must do something like prove that they are linearly independent, and no one who followed this approach did this, so I deducted a point.

    The nicest solution was to realize that $f = ax^3 + bx^2 + cx + d$ is a polynomial of degree 3, and to say that $T(f) = 0$ is to say that f has four zeros at 1, 2, 3 and 4. This is impossible unless f is the zero polynomial. Now the proof that a polynomial of degree n cannot have more than n zeros is not exactly trivial, and that's where Lagrange interpolation comes in; if you read the end of section 1.6 you'll see how Lagrange interpolation can be used to prove this; note in particular the very last sentence.

    (This paragraph is bonus material, and can be ignored if you're not interested.) In abstract algebra (Math 110 or 117), you'll learn a different way to prove that a polynomial over a field has at most n distinct roots. The basic idea is this. You can divide polynomials just like you can divide integers, and similar to integers there is a division algorithm that gives a quotient and remainder (you may have even done this in some previous math class). Now if a polynomial $p(x)$ of degree n has a root, say a, then if you divide $p(x)$ by $x - a$ you'll find the remainder must be 0, and so $p(x) = (x - a)q(x)$ for some polynomial $q(x)$ which must then have degree $n - 1$. By induction you find $p(x)$ has at most n distinct roots. If the field is $\mathbb{C}$, this is one half (the easy half) of the famous Fundamental Thereom of Algebra, which states that every polynomial over $\mathbb{C}$ of degree n has exactly n roots. The hard part is to show that it has a root at all.

15. One direction of this was already proved in problem 14(c) of Section 2.1, which you were assigned on previous homework. In fact this was a problem I graded and wrote solutions for. So this time I just graded the other direction, namely that if $T(\beta)$ is a basis for W then T is an isomorphism. I gave one point for a decent effort at solving the problem but only gave two points to those who had a solid proof.

    I realized partway through grading that the problem is actually not true as stated! You need to make some other assumption. One way to fix the problem is to assume that if $\dim V = n$, then $T(\beta)$ consists of n distinct elements. All of you guys made this assumption without really thinking about it, and that's fine. But it doesn't have to be true. I looked for errata for the textbook and found a list at `http://www.math.ilstu.edu/linalg/` (I've also posted a link to this page on my course webpage); there they fix the problem in a different way by stating that $\dim V = \dim W = n$. Of course these two fixes are equivalent.

    What is the problem? Well without one of these assumptions here is a counterexample. Let $V = \mathbb{R}^2$ and $W = R$, and define T so that $T(1,0) = T(0,1) = 1$; this extends to a unique linear transformation $T : \mathbb{R}^2 \to \mathbb{R}$ by the Universal Property (Theorem 2.6). Now $\beta = \{(1,0),(0,1)\}$ and $T(\beta) = \{1\}$ which is a basis for $\mathbb{R}$, but T is certainly not an isomorphism!

    There were several ways to solve the problem, involving more or less work depending on how many theorems you invoked. It's good to be familar with the theorems and invoke them (make sure you remember them for tests then!) but it's also good to be able to do the problem from scratch. Let me describe several approaches then. The first thing most people did was prove $\dim V = \dim W = n$ by assuming that $T(\beta)$ consists of n distinct elements; this of course follows immediately. Now at this point some people claimed that by Threrem 2.19 it follows T is an isomorphism. Not so! Theorem 2.19 simply says that V and W are isomorphic. It doesn't follow that just any old linear transformation between them is an isomorphism–there are plenty that aren't. We have to show T is in fact an isomorphism.

    There are three things to check: T is linear, 1-1, and onto. We're already given that it's linear. By Theorem 2.5 (a good theorem to remember), it's enough to just show one of 1-1 and onto (this only holds if $\dim V = \dim W$!). And in fact you can use Theorem 2.2 to immedately see that $R(T) = W$ and so T is onto; the result follows.

    Let me show how to do both 1-1 and onto from scratch as well, because that's useful to learn how to do and be comfortable with. Let $\beta = \{v_1, \ldots, v_n\}$. To show T is 1-1, we need to show that if $T(\sum a_i v_i) = 0$, then $a_i = 0$ for all i. But this follows since by linearity $T(\sum a_i v_i) = \sum a_i T(v_i)$, and the $T(v_i)$ are a basis of W and thus linearly independent, so if $\sum a_i T(v_i) = 0$ then all the $a_i = 0$.

    To show onto, we must prove that for all $w \in W$ there exists some $v \in V$ such that $T(v) = w$. But since the $T(v_i)$ form a basis for W, we can write $w = \sum b_i T(v_i)$. By linearity $w = T(\sum b_i v_i)$, so let $v = \sum b_i v_i$.

    A few people picked a basis γ for W and then invoked Thereom 2.6 to create a new linear transformation, which they happened to name T, to map β to γ. This is completely wrong–you are already given T so you can't just make up some new linear transformation and also call it T.

20. This was the hardest problem on the homework, and no one got it completely right although a few came close. The first difficulty is just to understand what the commutative diagram means; I won't try to explain that here. If you go on to study math in graduate school, you'll see plenty of commutative diagrams, including much bigger and more complicated ones, so it's not a bad idea to start to get comfortable with them now. The basic result that we need is that $L_a \phi_\beta = \phi_\gamma T$, where both sides are linear transformations from V to F^m.

    We want to show $\dim R(T) = \dim R(L_A)$. Now $R(T)$ is a subspace of W and $R(L_A)$ is a subspace of F^m. We need to compare these somehow. We use the fact that ϕ_γ is an isomorphism between W and F^m, and problem 17 (following the hint). Applying ϕ_γ to $R(T)$, we have that $\phi_\gamma(R(T))$ is a subspace of F^m (by 17(a)), and furthermore that $\dim R(T) = \dim \phi_\gamma(R(T))$ by 17(b). We will now show that $\phi_\gamma(R(T)) = R(L_A)$, so their dimenions are the same and therefore $\dim R(T) = \dim \phi_\gamma(R(T)) = \dim R(L_A)$.

    To show that two sets are equal, you can show that each is contained in the other. So let $z \in R(L_A)$. This means there exists some $y \in F^n$ such that $L_A(y) = z$. Since ϕ_β is an isomorphism, it is onto, which means there exists some $x \in V$ such that $\phi_\beta(x) = y$. This means $L_A(\phi_\beta(x)) = z$. But by commutativity of the diagram, $L_a \phi_\beta = \phi_\gamma T$, so $\phi_\gamma(T(x)) = z$ as well. But this means $z \in \phi_\gamma(R(T))$.

Conversely, let $z \in \phi_\gamma(R(T))$. This means there exists some $y \in R(T)$ such that $\phi_\gamma(y) = z$, which means there exists some $x \in V$ such that $\phi_\gamma(T(x)) = z$. Again using commutivity of the diagram we have $L_A(\phi_\beta(x)) = z$. Let $v = \phi_\beta(x)$; this means $L_A(v) = z$, so clearly $z \in R(L_A)$.

Now to prove the nullities are the same, you can do a similar argument. You might want to try it out; these kinds of arguments are very common when dealing with commutative diagrams. However most people who solved the problem noticed that you can save yourself the work by invoking the dimension theorem at this point–a very good observation. Since $\dim R(T) + \dim N(T) = \dim V = n = \dim F^n = \dim R(L_A) + \dim N(L_A)$ and we've just proven $\dim R(T) = \dim R(L_A)$, it follows immediately that $\dim N(T) = \dim N(L_A)$.

23. Like problem 15, this problem is also incorrect as stated, but in this case there doesn't seem to be any errata provided by the authors. This is is a more minor technical error, though. In Example 5 of Section 1.2, a sequence σ is defined as a function from $\mathbb{N}$ (the natural numbers $1, 2, \ldots$) to F, whereas in the problem they rely on $\sigma(0)$ being defined. Now you could try to get around this by defining $\sigma(0) = 0$, but in this case T won't be an isomorphism, as you can check. There are two ways to fix it. The first is to simply define $T(\sigma) = \sum_{i=0}^n \sigma(i+1)x^i$, which is probably what the text should have done. This is a little confusing, though. Less confusing is to simply allow, for this problem only, sequences to start from index 0. That means σ will be a function from $\mathbb{N} \cup \{0\}$ to F, or more concretely that the sequence will be labeled $a_0, a_1, a_2, \ldots$ instead of $a_1, a_2, \ldots$. If you're even more confused now, don't worry too much about it.

    To prove that T is an isomorphism, we have to prove that T is linear, 1-1, and onto. Linearity is just a straightfoward calculation, and I didn't check the details carefully for those who did it, but here's how it goes. Let σ and τ be two sequences in V, and let $a \in F$. Then $T(\sigma + \tau) = \sum_{i=0}^n (\sigma + \tau)(i)x^i \sum_{i=0}^n = (\sigma(i) + \tau(i))x^i = \sum_{i=0}^n \sigma(i)x^i + \sum_{i=0}^n \tau(i)x^i = T(\sigma) + T(\tau)$. Also $T(a\sigma) = \sum_{i=0}^n a\sigma(i)x^i = a\sum_{i=0}^n \sigma(i)x^i = aT(\sigma)$.

    Now to prove 1-1 and onto, there are two possibilities. One is to show that T has an inverse function (you must show that it's both a left and right inverse). The other is to show 1-1 and onto directly. I showed the former in section, but it seems people found that confusing, so perhaps you'd prefer the direct route. Here they both are.

    First to directly show that T is 1-1, we need to prove that if $T(\sigma) = 0$ (the zero polynomial), then $\sigma = 0$ (the zero sequence). The contrapositive is perhaps more straightforward here: Say $\sigma \neq 0$. Then there exists some k such that $\sigma(k) \neq 0$. But then $T(\sigma)$ has a term $\sigma(k)x^k$ which is non-zero, so it cannot be the zero polynomial.

    To show directly that T is onto, given any polynomial $\sum_{i=0}^n a_i x^i$, we must find some sequence $\sigma \in V$ such that $T(\sigma) = \sum_{i=0}^n a_i x^i$. But this is easy: Simply define $\sigma(i) = a_i$ for all $0 \leq i \leq n$ and define $\sigma(i) = 0$ for all $i > n$. Note that $\sigma \in V$ since only finitely many of its terms are nonzero.

    The other approach to showing T is an isomorphism is to construct an explicit inverse. This looks similar to the "onto" proof above. Let $U : W \to V$ be defined so that $U(\sum_{i=0}^n a_i x^i) = \tau$, where τ is a sequence defined so that $\tau(i) = a_i$ for all $0 \leq i \leq n$ and $\tau(i) = 0$ for all $i > n$. I claim $T^{-1} = U$; to prove this we need to show that TU is the identity of W and that UT is the identity of V.

    Now $TU(\sum_{i=0}^n a_i x^i) = T(\tau) = \sum_{i=0}^n \tau(i)x^i = \sum_{i=0}^n a_i x^i$ by the definition of τ. Also $UT(\sigma) = U(\sum_{i=0}^n \sigma(i)x^i) = \sigma$, since U constructs a sequence whose value at each i is exactly $\sigma(i)$. We've proven T is an isomorphism.