Math 115A
Homework 4 Comments

I graded 8 of the problems:
Section 2.2: 4, 8, 10, 13
Section 2.3: 2b, 4b, 11, 17

Each problem is worth 2 points. A grade of 0 indicates no solution or a substantially wrong solution. A grade of 2 indicates a correct or nearly correct solution. Otherwise the grade given is 1.

If you believe a problem was misgraded, or I made some addition or other error, please write a short note explaining the situation, attach it to your homework, and return it to me (either in person, in my mailbox, or under my office door). I'll take a look and afterwards leave your homework in a box outside my office.

The following are comments and occasionally solutions for the graded problems.

General Comments
Since I only graded 8 problems, the maximum number of points was 16. The high score was 14 and the median 11. This homework was a little shorter and easier than the previous ones, and I didn't grade many of the hardest problems (although I covered them in section or office hours), so the scores were pleasantly a little higher overall than usual.

2.2

4. Most people had no trouble with this. For each of the basis vectors $E_{ij} \in \beta$ you compute $T(E_{ij})$ and write the result in the coordinates of γ. You have to keep the order of both bases straight. The answer turns out to be
$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

8. I was pretty lenient with grading this, but a good solution would look like this. Writing vectors as row vectors to save space (they'd normally be written as column vectors), let $[x]_\beta = (a_1, \ldots, a_n)$ and $[y]_\beta = (b_1, \ldots, b_n)$. Then
$T(cx+y) = [cx+y]_\beta = (ca_1+b_1, \ldots, ca_n+b_n) = c(a_1, \ldots, a_n)+(b_1, \ldots, b_n) = c[x]_\beta+[y]_\beta = cT(x)+T(y)$.

10. Most people had no trouble with this (maybe because the answer is in the back of the book?). The ith column represents $T(v_i)$ written in the coordinates of β.

13. I did this in section and gave full credit if you just used my solution, but make sure you understand it. Here it is again. We want to show that if $aT + bU = T_0$, the zero linear transformation, then $a = b = 0$. We argue by contradiction, showing the other cases cannot occur. If $a \neq 0$ but $b = 0$, then $T = \frac{1}{a}T_0 = T_0$, which is not true by hypothesis. Similarly it can't be the case that $a = 0$ and $b \neq 0$. So consider the case $a \neq 0$ and $b \neq 0$. Then we have $aT = -bU$ which implies $\frac{-a}{b}T = U$. Since $T \neq T_0$, there exists some $x \in V$ such that $T(x) = w \neq 0$. Now $U(x) = \frac{-a}{b}w$, so by linearity $U(\frac{-b}{a}x) = w$. Therefore $w \in R(T) \cap R(U)$, contradicting our other hypothesis. It follows T and U must be linearly independent.

2.3

2. I just graded BC^t and CA in part (b), since those were not in the back of the book; one point for each correct answer. The solutions were $(12, 16, 29)^t$ and $(20, 26)$ respectively.

4. I just graded part (b), since it relied on 5b of section 2.2 which you were asked to do. I gave one point for a correct solution, and one point if you used Theorem 2.14. The main point of this problem was to make sure you understand what Theorem 2.14 is saying, namely that you can compute a linear transformation first and then write it in a basis, or convert the transformaion to a matrix and its argument into a column vector and multiply, and you'll get the same answer. This problem is asking you to do the latter. So if you did problem 5b of section 2.2, or looked in the back of the book, you found that
$$[T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Also $[f(x)]_\beta = (4, -6, 3)^t$. It follows from Theorem 2.14 then that

$$[T(f(x))]_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ -6 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 0 \\ 6 \end{pmatrix}.$$

11. There were not many really good proofs of this given. Here is what a good proof might look like.

    Proof. ($\Longrightarrow$) Let $y \in R(T)$. Then there exists some $x \in V$ such that $T(x) = y$. It follows $T(y) = T(T(x)) = T^2(x) = 0$ since $T^2 = T_0$ by hypothesis. It follows $y \in N(T)$, and so $R(T) \subseteq N(T)$.

    ($\Longleftarrow$) Let $x \in V$. Then $T(x) \in R(T) \subseteq N(T)$ by hypothesis, so $T^2(x) = T(T(x)) = 0$. Since this holds for all $x \in V$, it follows $T^2 = T_0$. □

17. Several people submitted a solution that I gather Prof. Krishna presented in office hours. What everyone submitted was nearly identical, and looked like it was a subset of his solution, perhaps what he wrote on the board. No explanation of anything was given, and in fact only two people bothered to define U, a linear transformation that suddenly appeared midway through. I could not make complete sense out of what anyone wrote, and I doubt they understood what they wrote either, so no one got credit for this.

    I also presented a solution to this problem in my own office hours which some people used, but again the solutions seemed mostly a copy of what I'd done without real understanding being shown, so I gave at most one point. I'd like to emphasize again that while it's fine to learn the solution from someone else, it's essential to go over it again on your own, make sure you understand it, and write it up in your own words. If you don't understand it then it's very unlikely someone reading your proof will understand what you've written either.

    There were really two parts to this problem. The first was to prove the hint. The second was to use the hint to determine which linear transformations T satisfy $T = T^2$. I assigned one point for each part.

    I assume what Prof. Krishna did in office hours was prove the hint. Here is my own proof which should be similar. Let $W = \{y : T(y) = y\}$. This is a subspace of V, as you should verify. We will show $V = W \oplus N(T)$, which means we have to show that $V = W + N(T)$ and $W \cap N(T) = \{0\}$. As in the hint, we note that $x = T(x) + (x - T(x))$ for any $x \in V$. Let $y = T(x)$ and $z = (x - T(x))$. We will show that $y \in W$ and $z \in N(T)$, which will prove $V = W + N(T)$. Now $T(y) = T(T(x)) = T(x) = y$ since $T^2 = T$, so $y \in W$. Furthermore $T(z) = T(x - T(x)) = T(x) - T^2(x) = 0$, so $z \in N(T)$.

    Now if $x \in W \cap N(T)$, then it means $T(x) = x$ and $T(x) = 0$, but that it is only possible for $x = 0$, so $W \cap N(T) = \{0\}$ and $V = W \oplus N(T)$.

    We still have to "determine all linear transformations $T : V \to V$ such that $T = T^2$". What does this mean? Well by Theorem 2.6 (the "Universal Property" of vector spaces) and its corollary, a linear transformation is determined completely by what it does to basis elements. Let α be a basis for $N(T)$, and extend to a basis β for V using the Replacement Theorem. Since we've proven $V = W \oplus N(T)$, it follows the elements $\beta \setminus \alpha$ are a basis for W. I don't know if you coverered this in class or not, but it takes some work to prove this. Another way to proceed is to let γ be a basis for W, and show that $\gamma \cup \alpha$ is a basis (call it β) for V since $V = W \oplus N(T)$; again the proof is not trivial. Note the similarity to midterm problem 3 and sample midterm problem 2.

    Now we can describe $T : V \to V$ more precisely. It is a linear transformation for which there exists a basis β of V such that for all $v \in \beta$, either $T(v) = v$ or $T(v) = 0$. Written in matrix form, $[T]_\beta$ looks like a diagonal matrix whose diagonal entries are either 1 or 0. Furthermore, given any linear transformation which satisfies these properties, it is easy to show that $T = T^2$. So we have determined these linear transformations completely.