Math 115A
Homework 3 Comments

I graded 10 of the problems:
Section 1.6: 14, 15, 31a
Section 2.1: 5, 9ae, 10, 14c, 15, 16, 22

Each problem is worth 2 points. A grade of 0 indicates no solution or a substantially wrong solution. A grade of 2 indicates a correct or nearly correct solution. Otherwise the grade given is 1.

If you believe a problem was misgraded, or I made some addition or other error, please write a short note explaining the situation, attach it to your homework, and return it to me (either in person, in my mailbox, or under my office door). I'll take a look and afterwards leave your homework in a box outside my office.

The following are comments and occasionally solutions for the graded problems.

General Comments
There was a big drop in quality for this set over the previous two. The median was 9 and the high score only 14. I think it's less a case of you guys not understanding the material so much as just not putting in the work necessary to do the problems correctly. This was a particularly long and difficult assignment, following two other long and tough assignments, so I can understand people getting worn out. Also probably midterms in other classes have started as well. Definitely watch out that you don't get behind, though–the pace of this class will probably stay very fast.
There were some problems where people seemed to have trouble; I've included solutions for those so please take a look at them.

1.6

14. Most people did fine on this and in fact chose the same bases for each space that I did, which made it easier to grade than I feared. For W_1 a basis is $\{(0,1,0,0,0),(0,0,0,0,1),(1,0,1,0,0),(1,0,0,1,0)\}$ and for W_2 a basis is $\{(1,0,0,0,-1),(0,1,1,1,0)\}$. The dimensions are thus 4 and 2 respectively.

15. Nearly everone got that the dimension was n^2-1 (in the back of the book), so that was a free point. Fewer were able to justify it, though. A basis contains E_{ij} for $i \neq j$, which contributes n^2-n elements, and $E_{ii}-E_{nn}$ for $1 \leq i < n$ which contributes $n-1$ elements. It is not hard to show that these are LI; you can argue that they must span simply because if not the dimension would be n^2 and clearly not every matrix has trace 0.

31. I just graded part (a). I didn't grade anything involving sums this time, but I learned from Prof. Krishna that he covered sum and direct sum in class and they may well appear on tests. This one was easy if you remembered some theorems in the book. Since $W_1 \cap W_2$ is a subspace of of W_2, then by Theorem 1.11 the result follows. I took off a point if you didn't reference the theorem or otherwise justify the result. Note that what's happening in Theorem 1.11 is that we start with a basis for $W_1 \cap W_2$ and then extend to a basis for W_2. Quite a few people tried to start with bases for W_1 and W_2 and argue about their intersection somehow, which unfortunately doesn't work.

2.1

5. There was a lot to do for this problem, and I concentrated mainly on arguments for null space and whether T is onto. For null space, almost nobody put in the work to show that it is $\{0\}$. Many simply claimed it, while others used the dimension theorem which you're not allowed to use since you're supposed to be verifying that it holds.

   Calculating the null space is not a completely trivial matter. You want to find all $f(x) \in P_2(\mathbb{R})$ such that $xf(x)-f'(x)=0$. At first glance this looks like a first-order differential equation! In fact one person tried to solve it that way. But what's important here (as in problems 15 and 16) is that we're only dealing with polynomials; the purpose of these problems are to help you get more comfortable with polynomials as vector spaces.

   Polynomials are a fairly concrete vector space; in fact we know a nice basis for $P_2(\mathbb{R})$ which is simply $\{1,x,x^2\}$. So any polynomial in $P_2(\mathbb{R})$ can be written as $f(x)=ax^2+bx+c$, where $a,b,c \in \mathbb{R}$. Since we can describe these vectors explicitly, it is now not so difficult to find the null space. We have $xf(x)-f'(x)=x(ax^2+bx+c)-(2ax+b)=ax^3+bx^2+(c-2a)x-b$, and if this equals 0 (meaning is the zero polynomial) then we must have $a=b=c=0$.

For onto, it was enough to argue that the rank of T is 3 and the dimension of $P_3(\mathbb{R})$ is 4, so T cannot be onto.

9. I just graded parts (a) and (e); one point each. Most people had no trouble with this.

10. Just about everyone calculated $T(2,3)$ correctly. Again the answer is in the back of the book, so I'm not sure if everyone really understood it, but the point is that $T(2,3) = 3T(1,1) - T(1,0)$ by linearity. Almost no one put in the work to really show T is 1-1, though. One way to show it is to argue that $(1,4)$ and $(2,5)$ are LI (clearly one is not a multiple of the other) and so form a basis for $\mathbb{R}^2$; it follows $R(T) = 2$ and by the dimension theorem $N(T) = 0$.

14. I just graded part (c), which is a bit more concrete than the previous two parts. A lot of people didn't even try this problem, but those who did were often successful. The easiest way to prove it is to use part (b) along with the dimension theorem. We know the dimension of V is n and since T is 1-1 we must have $N(T) = 0$ and thus $R(T) = n$. Since T is onto it follows the dimension of W is n. Now by part (b) we know the $T(v_i)$ are LI and thus must be a basis.

    It is also possible to argue this directly as follows. Since β is a basis of V we know any vector in V can be expressed as $\sum a_i v_i$ for $a_i \in F$. Now to show the $T(v_i)$ are LI, consider $\sum a_i T(v_i) = 0$; we want to show all $a_i = 0$. But by linearity we have $\sum a_i T(v_i) = \sum T(a_i v_i) = T(\sum a_i v_i) = 0$. Since T is 1-1 it follows $\sum a_i v_i = 0$, and since the v_i are LI it follows all the $a_i = 0$.

    To directly prove that the $T(v_i)$ span W, argue as follows. Since T is onto it follows that for any $w \in W$ there exists some $v = \sum a_i v_i$ such that $T(v) = w$. But $w = T(v) = T(\sum a_i v_i) = \sum a_i T(v_i)$ by linearity, and so the $T(v_i)$ span W.

    Some people referenced Theorem 2.6 (the "universal property" of vector spaces) for this problem. However it's not relevant here: Theorem 2.6 is about how if you have an arbitrary function from basis elements to some vector space, then it can be uniquely extended to a linear transformation. In this problem we are already given that T is linear. It is important to realize still that any linear transformation T is determined precisely by what it does to basis elements; that's part of what Theorem 2.6 is saying. See problem 22 below.

15. I debated about whether to grade both this and problem 16, since they are so similar. I thought everyone would either get both or none right. But I'm glad I did grade both because I found quite a few people had more trouble with this one than with problem 16, which I think reflects the fact that most people (myself included) are more comfortable with differentiation than with integration.

    Like problem 9, these problems were not so much a test of calculus ability as a test of how comfortable you are with polynomials as a vector space. Like problem 9, the best way to approach the problem is to recall you have a nice basis for $P(\mathbb{R})$ (the infinite basis $1, x, x^2, \ldots$), and therefore any polynomial can be written uniquely in the form $a_n x^n + \cdots + a_0$. Now you can explicitly integrate or differentiate this polynomial to prove the results. If you didn't do this in your homework (and only a couple people did) then I suggest you try it.

16. See the comments for problems 9 and 15.

22. One person did this problem perfectly correctly, but unfortunately no one else really came close. It's an important problem that both gives an idea geometrically of what a linear transformation must look like (as in problem 23) and presages section 2.2. As you guys must have covered section 2.2 in class before doing this homework I'm surprised more people didn't get this problem.

    The key to this problem is the fact that any linear transformation is determined completely by what it does to basis elements of the domain. It's important to understand this. In this case the domain is $\mathbb{R}^3$ and we know a good basis for $\mathbb{R}^3$, namely the standard basis. In fact in the problem our vector (x, y, z) (confusing since x, y, z are normally used for representing vectors, but here they are scalars) is implicitly written in terms of the standard basis. Now T must map e_1, e_2, e_3 to some elements of $\mathbb{R}$, so call them a, b, c respectively. It follows by linearity that for arbitrary $(x, y, z) = xe_1 + ye_2 + ze_3$, we have by linearity $T(x, y, z) = xT(e_1) + yT(e_2) + zT(e_3) = ax + by + cz$.

    Once you understand this part it should be easy to generalize to the other cases, which I'll leave as an exercise. Note that for $T : F^n \to F^m$, each e_i is mapped to a vector, call it $v_i \in F^m$, and so what you want to prove is that $T(\sum a_i e_i) = \sum a_i v_i$.