# Calculus Theory Challenge Problems

Fall 2010

September 15, 2010

## Instructions

When writing a solution, indicate both the chapter number (or "Misc" for miscellaneous problems) and the problem number (the problem number on this sheet, not the Spivak problem number). For multi-part problems, it is fine to solve just some of the parts. The number of points each problem or part is worth is given in square brackets; e.g. "[5]" means 5 points. A problem of the form "Spivak $x$" means it is from the text (4th edition), problem $x$ in the appropriate chapter.

Your handin must include on the first page a table with the number of each problem and a second column for me to record your score for that problem.

Calculus Theory 1 students may get credit for any of the miscellaneous problems and Spivak chapter 1 to 12 problems. Calculus Theory 2 students may get credit for any of the miscellaneous problems (not attempted last year) and Spivak chapter 13 to 30 problems. Calculus Theory 2 students may be able to get credit for chapter 1 to 10 problems as well, but generally only the harder problems, say worth 20 points (per part) or more. See me if you want to get credit for any of these. You are of course welcome to work on any problem you like for no extra credit!

You may also come up with your own problems, and explore any question you have particular interest in. Talk with me before you turn in anything, however, so I can give some guidance and ensure the problem is appropriate. Points will be awarded depending on the quality of your work.

There may also be variable point point problems in which you can get more points by going farther than simply providing the minimal solution required.

There are no required number of points of challenge problems for the fall. I may require a number of points in the spring, and problems you do in the fall will count toward the spring total. Any points you get over the required points will count as extra credit, applied in the spring. The exact formula for extra credit is still to be determined.

## Miscellaneous Problems

1. Show that there is a 1-1 correspondence between fractional linear transformations $\frac{ax+b}{cx+d}$ and $2 \times 2$ matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ as follows. Let $f(x) = \frac{ax+b}{cx+d}$, $g(x) = \frac{Ax+B}{Cx+D}$, $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $N = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. If $f(g(x)) = \frac{\alpha x+\beta}{\gamma x+\delta}$, then show that $MN = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. [15]

2. Let
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
Both of these matrices are in $SL_2(\mathbb{Z})$, which is the special linear group of $2 \times 2$ matrices with integer coefficients which have determinant 1. Prove that any matrix in $SL_2(\mathbb{Z})$ is generated by $S$ and $T$ (meaning it is a matrix product of a number of $S$, $T$, $S^{-1}$ and $T^{-1}$ matrices). Hint: Start with a matrix $M$ in $SL_2(\mathbb{Z})$ and see if you can get the identity matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ from $M$ by multiplying it repeatedly (on the left or right as necessary) by $S$, $S^{-1}$, $T$ and/or $T^{-1}$. [50]

3. Find two functions $f$ and $g$ that are not constant multiples of each other that satisfy $f' = g$ and $g' = f$. [30]

4. Find a function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(f(x)) = -x$ for all $x \in \mathbb{R}$. [50]

5. Prove that the derivative of an odd function is an even function, and vice-versa. [5]

6. (Emily Steemers) Find a continuous function with the following prooperty: There are two points on its graph such that the tangent line at one point intersects the second point and is perpendicular to the tangent line at the second point. Or prove such a function does not exist. [variable, but at least 10]

# Chapter 1

1. Find a subfield of $\mathbb{R}$ that is not $\mathbb{R}$ or $\mathbb{Q}$ and prove it is a subfield (meaning that it contains 0 and 1 and is closed under addition, multiplication, and taking inverses). [20]

# Chapter 2

1. Spivak 3(e). (Each part is worth [5].)

2. Spivak 27 and 28 (must be done together). [50]

# Chapter 3

1. Spivak 11. ([5] each for parts (a)-(c); [10] each for (d) and (e))

2. Spivak 17. ([5] for each part)

3. Spivak 18. [20]

4. Spivak 19. ([10] per part)

5. Spivak 21. ([3] per part)

6. Spivak 22. ([5] per part)

7. Spivak 23. ([5] per part)

8. Spivak 24. ([10] per part)
9. Spivak 25. [15]
10. Spivak 26. [10]
11. Spivak 27. ([5] per part)
12. Spivak 28. ([5] per part except (c) which is worth [10])

# Chapter 4

1. Spivak 7. ([5] per part)
2. Spivak 8. ([5] per part)
3. Spivak 17. ([3] per part)
4. Spivak 18. ([3] per part)

# Chapter 4, Appendix 1 (Vectors)

1. Spivak 1. ([5] per part)
2. Spivak 2. ([5] per part)
3. Spivak 3. ([5] per part)
4. Spivak 4. [10]
5. Spivak 5. ([5] per part)
6. Spivak 6. [10]
7. Spivak 7. [10]

# Chapter 4, Appendix 3 (Polar Coordinates)

1. Spivak 1. [10]
2. Spivak 2. [10]
3. Spivak 3. ([3] per part)
4. Spivak 4. ([3] per part)

# Chapter 5

1. Spivak 3(ii–vi). ([10] per part)
2. Spivak 4. ([2] per part)
3. Spivak 7. [10]
4. Spivak 8. ([5] per part)
5. Spivak 9. [5]
6. Spivak 10. ([10] per part)
7. Spivak 15. ([5] per part)
8. Spivak 16. ([10] per part)
9. Spivak 17. ([5] per part)
10. Spivak 18. ([10] per part)
11. Spivak 19. [10]
12. Spivak 20. [15]
13. Spivak 21. ([5] per part)
14. Spivak 22. [10]
15. Spivak 25. ([3] per part)
16. Spivak 26. ([10] per part)
17. Spivak 27. ([3] per part)
18. Spivak 31. [15]
19. Spivak 34. [5]
20. Spivak 35. ([5] per part)
21. Spivak 36. ([5] per part, including the definition)
22. Spivak 38. ([5] per part)

# Chapter 6

1. Spivak 6. ([5] per part)
2. Spivak 9. ([10] per part)

# Chapter 7

1. Spivak 7. [5]
2. Spivak 12. ([5] per part)

# Chapter 8

1. Spivak 3. ([10] for (a), [20] for (b))
2. Spivak 11. ([5] for each (a)–(c), [20] for (d))

# Chapter 9

1. Spivak 13. [15]
2. Spivak 14. [10]
3. Spivak 15. ([10] per part)
4. Spivak 16. [10]
5. Spivak 19. ([10] per part)
6. Spivak 20. ([10] per part)
7. Spivak 21. ([3] per part)
8. Spivak 23. [5]
9. Spivak 24. [5]
10. Spivak 25. [5]
11. Spivak 26. ([2] per part)
12. Spivak 27. [15]
13. Spivak 28. ([10] per part)
14. Spivak 29. [15]
15. Spivak 30. ([2] per part)

# Chapter 10

1. Spivak 2. ([3] for each part)
2. Spivak 11. [15]
3. Spivak 13. ([5] for (a), [10] for (b))
4. Spivak 17. [10]
5. Spivak 18. ([3] for (a) and (b), [5] for (c))
6. Spivak 19. ([10] per part)
7. Spivak 20. [20]
8. Spivak 23. ([5] per part)
9. Spivak 25. [5]
10. Spivak 26. ([10] per part)
11. Spivak 27. ([10] per part)
12. Spivak 28. [10]
13. Spivak 29. [15]
14. Spivak 30. [20]
15. Spivak 31. [10]
16. Spivak 32. ([10] per part)

# Chapter 11

1. Spivak 4. ([3] for (a); [10] for (b) and (c))
2. Spivak 6. [10]
3. Spivak 8. ([10] per part)
4. Spivak 14. [15]
5. Spivak 15. [15]
6. Spivak 17. [10]
7. Spivak 18. [20]
8. Spivak 19. ([8] per part)
9. Spivak 20. [20]
10. Spivak 21. [2]

11. Spivak 22. [5]
12. Spivak 25. ([5] per part)
13. Spivak 26. [10]
14. Spivak 38. [10]
15. Spivak 39. [10]
16. Spivak 41. ([10] per part)
17. Spivak 57. [10]
18. Spivak 65. [10]
19. Spivak 66. ([5] per part)

# Chapter 12

1. To be determined.