Extra Credit: Classification of Quadratic Equations in Two Variables

Algebra 2 Honors

May 4, 2011

Note: This extra credit is worth up to 2% of your semester grade (0.2% for question 1 and 0.3% for each of the other 6 questions). You may discuss the problems with your classmates and myself (and I encourage you to do so), but your written solutions must be your own work. Solutions are due Friday, May 20.

A *quadratic equation in two variables* is an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where A, B, C, D, E, F are all real numbers. Some examples are $x - y = 0$ (the line $y = x$), $3x - y + 4 = 0$ (the line $y = 3x + 4$), $x^2 + y^2 - 4 = 0$ (the circle $x^2 + y^2 = 4$), $-x^2 + y = 0$ (the parabola $y = x^2$), $4x^2 + 9y^2 - 36 = 0$ (the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$), $xy - 1 = 0$ (the hyperbola $xy = 1$), and so forth.

A *conic section* is the curve that results when you intersect a cone (actually a sort of "double cone") and a plane. Look up conic section on the internet to see good pictures. We have been studying conic sections the last two years: parabolas, ellipses, and hyperbolas. Circles can be thought of as special cases of ellipses. There are also "degenerate" conic sections: a point, a line, and two intersecting lines.

Mathematicians love to classify things. One very interesting classification is that of the quadratic equations in two variables. It is possible to prove, as the examples above might have hinted at, that every such equation (save for some special cases) defines a conic section! The exercises in this handout will guide you through the process. The strategy is to sytematically examine all cases, in particular those in which particular constants A, B, C, D, E, F are zero.

1. If every constant is zero, we get $0 = 0$ which is satisfied by every point (x, y). This is one of the special cases that is not a conic section. The next simplest case is that in which exactly one of A, B, C, D, E, F is nonzero, but all of the others are zero. For example if $A \neq 0$ but $B = C = D = E = F = 0$, then we get $Ax^2 = 0$, and since $A \neq 0$ we can divide both sides by A to get $x^2 = 0$. This is true if and only if $x = 0$, and this is the equation of a line (the y axis, in fact), which is a degenerate conic section. Look at each of the other 5 cases in which exactly one of the constants is non-zero, and describe the curve that results.

2. Let's now look at the "linear" case in which only only degree 1 or degree 0 terms appear. In other words, $A = B = C = 0$ and we are left with $Dx + Ey + F = 0$. We can assume that at least two of D, E, F are non-zero since we handled the other cases in the previous problem. Show that all of the curves that result in this case are lines.

3. Now let's look at some special cases which involve terms with degree 2. First assume $B = D = E = 0$ which leaves the equation $Ax^2 + Cy^2 + F = 0$. Again at least two of A, C, F can be assumed to be non-zero. Show that in this case you get either an ellipse (including the special case a circle), a hyperbola, a point (which can be thought of as a degenerate circle!), zero points, two points or two lines.

4. Now let's make things harder and consider the case in which $A \neq 0$ but $B = C = 0$. So we have $Ax^2 + Dx + Ey + F = 0$ with $A \neq 0$, and at least one of D, E, F is also non-zero. Show that the solution either either nothing, one point, two points, or a parabola. (Hint: You'll want to complete a square in one case.) Also consider the symmetric case $Cy^2 + Dx + Ey + F = 0$ where $C \neq 0$ and at least one of D, E, F is also non-zero.

5. We're ready for the next harder case! This is $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where A and C are both non-zero. Show that the solution is either nothing, a point, an ellipse or a hyperbola. Again you'll want to complete the square, maybe even twice!

6. We have now covered all cases in which $B = 0$. Notice that in every case the major and minor axes of the conic sections are aligned with the x and y axes. When $B \neq 0$ this is no longer the case. As you might guess a rotation is involved. Let's explore first the classic case $xy - 1 = 0$. First plot the hyperbola $xy = 1$ to see what it looks like. Notice that the axes seem to have been rotated 45 degrees. Now change coordinates using the linear transformation

$$\begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}$$

and show that the transformed equation is a hyperbola in standard form. Explain what is going on.

7. Now that we have the basic idea, we're ready to handle the general case $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Use a rotation by angle θ to change coordinates and eliminate the xy term, so that the result after rotation has the form $A'x^2 + C'y^2 + D'x + E'y + F' = 0$. Define θ precisely in terms of A, B, C, D, E, F, and then define each of A', C', D', E', F' precisely in terms of A, B, C, D, E, F and θ. You have now used the famous mathematical trick of "reducing to a previous case"; namely to one of the cases we have already considered. Explain why you have solved the problem.