Complex Numbers, Polynomials, and Symmetry

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March 9, 2011

1 Roots of Polynomials

The complex numbers arise naturally from attempts to find roots of polynomials. A *polynomial* is an expression such as $3x^4 + \pi x^3 + x^2 + 5x + \sqrt{2}$ in a single variable (in this case x) that consists of a sum of terms in which x appears with various whole-number powers. The highest power of x is called the *degree* of the polynomial. So the previous polynomial has degree 4.

A *root* of a polynomial is a value for x that makes the value of the entire polynomial zero. For example consider the degree 2 polynomial $x^2 - 5x + 6$. If $x = 2$ then the polynomial evaluates to $2^2 - 5 \cdot 2 + 6 = 0$, so 2 is a root of the polynomial. Likewise if $x = 3$ then the polynomial evaluates to $3^2 - 5 \cdot 3 + 6 = 0$, so 3 is also a root of the polynomial. This should not be surprising since $x^2 - 5x + 6 = (x - 2)(x - 3)$, so by the ZPP (zero product property) the polynomial is zero iff (if and only if) $x - 2 = 0$ or $x - 3 = 0$; in other words iff $x = 2$ or $x = 3$. You learned this back in Algebra 1.

Now consider the polynomial x^2+1. This polynomial has no real roots because if $x^2+1 = 0$ then $x^2 = -1$, but any real number squared must either be zero or a positive number. We could just give up and say there are no roots, but mathematicians don't like to give up. Instead we will define a new number, the "imaginary" number i, to be a square root of -1. Since $i = \sqrt{-1}$, it follows $i^2 = -1$ and so i is a root of $x^2 + 1$. Note that $-i$ is also a square root of -1 and also satisfies $(-i)^2 + 1 = 0$ so it is also a root of $x^2 + 1$, which therefore has two roots. You can check that $x^2 + 1 = (x + i)(x - i)$.

1. What is i^{2011}?

2. [adapted from Exeter Math 4C, August 2009, 12:2] What is wrong with the following argument?

$$i^{83} = (\sqrt{-1})^{83} = ((-1)^{1/2})^{83} = ((-1)^{83})^{1/2} = (-1)^{1/2} = \sqrt{-1} = i$$

3. Is i positive or negative? Is it possible to tell?

2 Algebra of Complex Numbers

Now that we have a new number i, let's see what we can do with it. First of all we can form a *complex number* $a + bi$ where a and b are real numbers. a is called the *real part* and b is called the *imaginary part* of the complex number.

1. How would you add complex numbers $2 + 3i$ and $4 + 5i$? Formulate a general rule for adding complex numbers $a+bi$ and $c+di$. Verify that your rule satisfies the associative and commutative laws of addition.

2. What is the additive inverse of $2 + 3i$? That is, what number $c + di$ can you add to it to get the additive identity (zero)? In general what is the additive inverse of $a + bi$?

3. How would you multiply complex numbers $2+3i$ and $4+5i$? Formulate a general rule for multiplying complex numbers $a + bi$ and $c + di$. Verify that your rule satisfies the associative and commutative laws of multiplication.

4. Does the distributive law hold? That is, does $(a+bi)[(c+di)+(x+yi)] = (a+bi)(c+di) + (a + bi)(x + yi)$?

5. Can we divide $1 + 2i$ by $3 + 4i$ and get a complex number? That is, does there exist some real c and d such that $\frac{1+2i}{3+4i} = c + di$? There is a nice trick to make this work.

6. Now find the multiplicative inverse of $2 + 3i$; that is, the number $c + di$ such that $(2 + 3i)(c + di) = 1$. Describe the multiplicative inverse of $a + bi$.

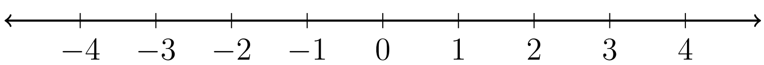
7. Is every real number also a complex number? Why or why not?

In the above exercises we have shown that complex numbers can be added and multiplied, and that they have additive and multiplicative inverses. They satisfy associative, commutative and distributive laws. A set of numbers that satisfies all of these properties is called a *field*. Fields you already know are the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$. We have now defined a new field $\mathbb{C}$ to be $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$, where i satisfies $i^2 = -1$.

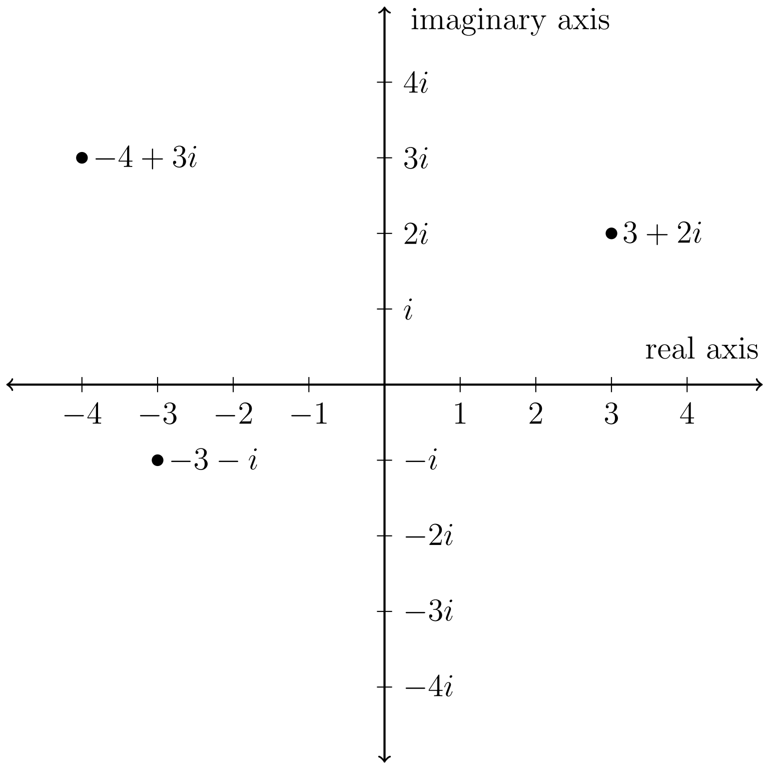
We will now examine just a few of the amazing properties of the field of complex numbers.

3 Geometry of Complex Numbers

One of the beauties of complex numbers is that they are so easy to visualize. You may remember in elementary school you visualized the real numbers using a number line.



We can visualize the complex numbers using two perpendicular number lines, one for the real part and one for the imaginary part. They intersect at zero, the origin. These are just like the x and y axes of the Cartesian coordinate system. Here they are called the *real axis* and the *imaginary axis*. So the complex number $x + iy$ can be plotted as a point (x, y) on a plane, which is called the *complex plane*. Note that while the real numbers are "one-dimensional", the complex numbers are "two-dimensional".



1. Plot the numbers $2 + 3i$ and $2 - 3i$ on the complex plane. These two numbers are complex conjugates. The *complex conjugate* of $z = a + bi$ is defined to be $\bar{z} = a - bi$. Where did you see a complex conjugate in the previous section? What symmetry relates a number and its complex conjugate?

2. We measure the "size" of a real number by using absolute value: $|x|$ is the distance from the real number x to the origin. In the same way we define the absolute value of a complex number to be its distance from the origin. This distance will always be a real number. What is $|3 + 4i|$? In general what is $|a + bi|$? What is $|a - bi|$? What do complex conjugates have to do with absolute value?

3. We can also view the complex numbers as a two-dimensional real vector space. Just think of $x + iy$ as the vector $[x, y]$. Look back in the handout *Basics of Linear Algebra* at the definition of a *basis* of a vector space. What would be the standard basis of $\mathbb{C}$ as a real vector space? What is the length of vector $[x, y]$?

4. Plot the set of points $\{\cos\theta + i\sin\theta : 0 \leq \theta < 2\pi\}$. What familiar object is this?

5. If complex number z is written as $z = x + iy$ then we say that $x + iy$ is the *rectangular form* of z. We can also write z in *polar form*! Let r be the distance from z to the origin, and let θ be the angle (in radians, of course!) that the line through the origin and z makes with the positive real axis. Then we can write z in either rectangular coordinates (x, y) or polar coordinates (r, θ). Give formulas for r and θ in terms of x and y, and vice-versa.

6. Just like we can write $z = x + iy$, we would like to be able to write "$z =$ (something in polar form)" so that we can manipulate it algebraically. Do this using r and $\cos\theta + i\sin\theta$ and show what this means geometrically.

7. Write the following numbers in polar form: (a) 1, (b) i, (c) $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$, (d) $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$.

4 Euler's Formula

In the usual high school coverage of complex numbers, one abbreviates $\cos\theta+i\sin\theta$ as $\operatorname{cis}(\theta)$. You never see this abbreviation after that, however, so let's use the correct notation from the start. *Euler's formula* states that

$$e^{i\theta}=\cos\theta+i\sin\theta.$$

The reason this notation is not normally shown in high school should be clear. We are using a number e that we haven't defined and whose definition is difficult, and we're using complex exponentiation which we also haven't defined. But I think it's still worth learning this notation now. We will simply define $e^{i\theta}$ to be $\cos\theta+i\sin\theta$ and show that the laws of exponents work the way we expect. If you take Calculus Theory, everything will be developed carefully and Euler's formula will be proven at the end of the second year.

1. Say that you start with a dollar and earn 100% (1 as a fraction) interest over one year. Of course you end with 2 dollars. Next say that you start with a dollar and earn 50% (1/2 as a fraction) interest each half year. Write the formula for how much you end up with after a year in this case and show that the answer is \$2.25. Now say that you start with a dollar and earn $\frac{100}{n}\%$ ($1/n$ as a fraction) interest each $1/n$ year. Write a formula for how much you end up with after a year in this case. Notice that as n gets bigger the amount you end up with seems to converge to a value between 2 and 3. By using your calculator and plugging in large values of n, determine this value accurately to three digits after the decimal place. The exact number you get as n goes to infinity is e.

2. Using Euler's formula, write the complex conjugates of the four numbers in exercise 7 of the previous section in polar form $re^{i\theta}$.

3. Plot the set of points $\left\{e^{i\theta}:0\leq\theta<2\pi\right\}$. What is $\left|re^{i\theta}\right|$?

4. Every complex number except zero can be written in polar form $re^{i\theta}$ with $r>0$, and we will restrict r to always be non-negative. Even in this case, there are still infinitely many polar forms for a single complex number. Describe all polar forms of the number $\sqrt{3}-i$. How might you restrict the angle to get a unique "canonical" value?

5. Describe all polar forms of the real number -1.

6. If $z=re^{i\theta}$, what is the complex conjugate $\bar{z}$ in polar form?

7. Polar form is especially useful when multiplying numbers. Multiply $(a+bi)(c+di)$ in rectangular form. Now multiply $(re^{i\alpha})(se^{i\beta})$ in polar form, assuming laws of exponents work as you would expect. Which is easier?

8. Let's make sure the laws of exponents really do work. Using Euler's formula, prove that $e^{i\alpha}e^{i\beta}=e^{i(\alpha+\beta)}$. You will need to recall your addition laws for sine and cosine that we derived earlier using matrices and other techniques. Rederive them if you've forgotten them.

9. Not only is multiplication easier using polar form, but it's also easy to understand what effect multiplication has geometrically. Plot the points $\sqrt{3} - i$ and $i(\sqrt{3} - i)$. What effect did multiplying by i have geometrically? Convert both $\sqrt{3} - i$ and i to polar form, multiply them, and see if you can explain what is going on.

10. Given a complex number z, what is the geometric effect of multiplying z by -1? What is the geometric effect of multiplying z by $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$? What is the geometric effect of multiplying z by $3i$?

11. Given a complex number z, what is the geometric effect of multiplying z by $re^{i\theta}$?

12. If multiplication is easier in polar form, then exponentiation should be easier still! Let $z = 1 + i\sqrt{3}$ and calculate z^4 in two ways. First use rectangular form, multiplying everything out by hand. Next convert to polar form, exponentiate in polar form, and then convert back to rectangular form. Which was easier?

13. We can now do even more amazing things. Calculate i^i. You may be very surprised at the answer! Use your calculator to find a rough numerical approximation of its value.

14. Find a special angle θ that when you plug it into Euler's formula, results in an equation relating the five most important constants in mathematics: e, π, i, 0, and 1. This has been called the most beautiful equation in all of mathematics.

5 Roots of Unity

Solutions to the equation $x^n - 1 = 0$ are called *roots of unity* since they are solutions to $x^n = 1$, or in other words the nth roots of one ("unity"). When we allow complex numbers, there are always exactly n distinct roots of unity, and they have a particularly simple and beautiful symmetry.

1. The single first root of unity is 1, and the two second roots of unity are 1 and -1. What are the four 4th roots of unity? Plot them on the complex plane.

2. Show algebraically that $x^4 - 1 = (x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$ where $\alpha, \beta, \gamma, \delta$ are the fourth roots of unity you found in the previous exercise. Express the right hand side of the equation as a product of three polynomials with all real coefficients.

3. What are the three 3rd roots of unity? Describe them both in rectangular form and polar form, and plot them on the complex plane.

4. Show algebraically that $x^3 - 1 = (x - \alpha)(x - \beta)(x - \gamma)$ where α, β, γ are the third roots of unity you found in the previous exercise.

5. What are the six 6th roots of unity? Describe them both in rectangular form and polar form, and plot them on the complex plane.

6. What is $\sqrt{i}$ and how many numbers z are there such that $z^2 = i$? Express them in polar form. Are they nth roots of unity? If so, what is n?

7. In general what are the nth roots of unity in polar form? What symmetry do they have?

6 Roots of Numbers

Now that we understand roots of unity, let's apply our knowledge to understand better the roots of arbitrary numbers.

1. What is $\sqrt{64}$? Which real numbers x satisfy $x^2 = 64$? Which complex numbers z satisfy $z^2 = 64$?

2. What is $\sqrt[3]{64}$? Which real numbers x satisfy $x^3 = 64$? Which complex numbers z satisfy $z^3 = 64$?

3. Let x be a positive real number. If we only allow real solutions, what is the difference between taking the nth root of x when n is even and when n is odd?

4. Look back at Exeter problem 46:2. Now that you have learned complex numbers, verify that $(-8)^{1/3}$ has three values (one real), that $(-8)^{1/4}$ has four values (all non-real) and that $(-8)^{2/6}$ is no longer ambiguous. List the 3rd and 4th roots of -8 in polar form.

5. In general, if we are given $z = re^{i\theta}$ in polar form, what are the nth roots of z in polar form? What do they look like if you plot them on the complex plane?

6. Let z be a complex number which is not real. Is it possible for $z^{1/n}$ to be real for some $n \in \mathbb{N}$? If so give an example. If not, explain why not.

7 Roots of Polynomials Revisited

1. Show that $x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$. What does this say about the roots of the polynomial?

2. Let $f(x)$ and $g(x)$ be polynomials. For example we could have $f(x) = x^3 - 6x^2 + 11x - 6$ and $g(x) = x^2 - 5x + 6$. If we can write $f(x) = (x-a)g(x)$ then clearly a is a root of $f(x)$ since $f(a) = (a-a)g(a) = 0$. In our example, $x^3 - 6x^2 + 11x - 6 = (x-1)(x^2 - 5x + 6)$. Verify that this is true. Then find two polynomials $h(x)$ and $k(x)$ such that $x^3 - 6x^2 + 11x - 6 = (x-2)h(x)$ and $x^3 - 6x^2 + 11x - 6 = (x-3)k(x)$.

3. It turns out that the converse is also true: If a is a root of polynomial $f(x)$, then there exists some polynomial $g(x)$ such that $f(x) = (x-a)g(x)$. That is, we can always "factor out" the linear term $(x-a)$ from $f(x)$ if a is a root. The proof (which is done early in Calculus Theory) is not too difficult and uses the *division algorithm*, which is just like long division of integers that you perhaps did in elementary school.

   For now, let's assume this is true and see how to use it. Verify that $x = 1$ is a root of $x^3 + x^2 - 17x + 15$ and see if you can figure out how to divide $x^3 + x^2 - 17x + 15$ by $x - 1$ using long division to get a degree 2 polynomial. Then factor the degree 2 polynomial to find the other two roots. We'll go over how to divide polynomials in class.

4. Notice that $x^2 - 2x + 1 = (x-1)(x-1)$. Therefore 1 is the only root of the polynomial, but in essence it is the root twice. We say that the the root 1 has *multiplicity* 2. In general if a is the root of polynomial $f(x)$, then a has multiplicity n if $(x-a)^n$ divides $f(x)$ (with no remainder) but $(x-a)^{n+1}$ does not divide $f(x)$. What is the multiplicity

of the root 1 in the polynomial $x^4 - 6x^3 + 12x^2 - 10x + 3$? What are the other roots of this polynomial and their multiplicities?

5. Factor $x^4 - 5x^2 + 6$ by hand and find its four roots.

6. Factor x^4+4x^2+4 by hand and find its four roots. Note that the roots have multiplicity 2 this time! Also note that they are imaginary!

7. There are ways to find roots quickly without resorting to the quadratic and cubic formulas. If $f(x)$ is a degree 2 polynomial and $f(x) = (x-a)(x-b)$, then we know $f(x) = x^2 - (a+b)x + ab$. That is, the coefficient of x must be the negative of the sum of the two roots, and the constant coefficient must be the product of the two roots. Use this method to quickly factor $x^2 + 2x - 35$. (This should be review.)

8. You can do the same trick for degree 3 and higher polynomials, although it gets harder. If $f(x)$ has degree three and $f(x) = (x-a)(x-b)(x-c)$, then $f(x) = x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc$. Explain the patterns and symmetries you see in this expression. Then see if you can use it to find the roots of $x^3 - 2x^2 - 5x + 6$.

8 The Fundamental Theorem of Algebra

We needed complex numbers to solve $x^2+1=0$. What happens now if we allow polynomials to have complex coefficients? That is, if we want to find the roots of an equation like $(1+2i)x^2+(3+4i)x+(5+6i) = 0$, do we need some kind of new "super complex" numbers? Happily the answer is no: Complex numbers are enough!

The *fundamental theorem of algebra*, one of the greatest results in mathematics, states that any polynomial of degree n with complex coefficients has exactly n complex roots, counting multiplicity. There are many proofs of this theorem (in fact there is a college undergraduate textbook titled *The Fundamental Theorem of Algebra* which gives ten different proofs) but none of them are easy. If you take Calculus Theory you will learn how to prove some important special cases of this remarkable theorem.

Here we prove the very special case of $n = 2$ by simply re-deriving the quadratic formula!

1. Review the derivation of the quadratic formula from the fall final exam. Show that if $ax^2 + bx + c = 0$ where a, b, c can now be complex numbers ($a \neq 0$), then it still holds that $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

2. Use the quadratic formula to find the two roots of $(1+2i)x^2+(3+4i)x+(5-4i) = 0$. Write them in rectangular form.